Completeness in Discrete-Time Process Algebra

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Abstract

We prove soundness and completeness for some ACP-style concrete, relative-time, discrete-time process algebras. We treat non-delayable actions, delayable actions, and immediate deadlock. Basic process algebras are examined extensively, and also some concurrent process algebras are considered. We conclude with $\mathrm{ACP_{drt}}$, which combines all described features in one theory.

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Contents

1 Introduction

1.1 Discrete-Time Process Algebras

Since [14] appeared, the ACP framework of process algebras has been extended with, among many other things, discrete-time extensions. Papers describing such extensions are for example [8, 10, 11].

With the recent appearance of [10, 13] discrete-time process algebra seems to have reached a decent state of maturity, and we felt now was the time to write a paper about the soundness and completeness issues involved with discrete time. To our knowledge, no such results have been published in the context of ACP before. And although we never really doubted the soundness and completeness of the respective theories, we felt that it would not hurt to prove these beliefs explicitly. And rightly so: it turned out that the axiomatizations we started out with were neither sound nor complete.

1.2 Soundness and Completeness

In this paper we will give elimination, soundness, and completeness results for several discrete-time process algebras. We restrict ourselves to concrete process algebras (i.e. without abstraction, without a silent step τ, and without an empty step ε), relative time, closed terms (i.e. no ω-completeness), and mostly basic process algebras (i.e. without merge operators). We do treat delayable actions and immediate deadlock.

In our definitions and notations we try to conform to [11, 13]. We use term deduction system semantics in the style of Section 2.2.3 of [13].

In the proofs of elimination theorems we make abundant use of term rewriting analysis. For further details on these techniques, especially the lexicographical path ordering, see [19]. In proving completeness we sometimes make use of Verhoef's General Completeness Theorem. For more information on this, see [25].

As the definitions and notations used for discrete-time process algebras have been subject to vehement revision over the past few years, we have compiled an appendix that tries to shed some light on these matters. Please study this Appendix A if you are not completely familiar with discrete-time ACP. Then, in Appendix B, we give a concise summary of the most important results presented in this paper.

Finally, the actual theorems and proofs on soundness and completeness are given in Section 2 (for process algebras that do not contain a merge operator) and Section 3 (for the ones that do).

2 Basic Process Algebras

2.1 Introduction

In this section we prove soundness and completeness for some discrete-time basic process algebras, i.e. not containing a merge operator. But we start by giving soundness and completeness theorems for two untimed basic process algebras, namely BPA and BPA_δ.

The purpose of giving these results about untimed algebras is twofold. First we want to show the principles our proof techniques are based on, and these simple process algebras are more suited to this purpose than the timed versions that will follow. Secondly, we felt that soundness and completeness proofs given in the literature are often a bit twisted, and sometimes even plain wrong.

After these untimed basic process algebras, by adding the time-unit delay, we proceed to the most simple timed one: $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta$. Then we add to this algebra undelayable deadlock, delayable actions, and finally immediate deadlock. The section culminates in soundness and correctness theorems for a basic process algebra that combines all the above extensions at the same time: $\mathrm{BPA}_{\mathrm{drt}}$.

2.2 Soundness and Completeness of BPA

Remark 2.2.1 (Alphabet)
For this section, and all sections to come, we presume the existence of a fixed, finite alphabet A, that can be considered a parameter of the respective theories. Furthermore, we define A_δ as $A \cup \{\delta\}$ and A_σ as $A \cup \{\sigma\}$, where δ and σ are still to be treated symbols that are not contained in A.

Definition 2.2.2 (Signature of BPA)
The signature of BPA consists of the *atomic actions* $\{a | a \in A\}$, the *alternative composition operator* $+$, and the *sequential composition operator* $\cdot$.

Remark 2.2.3 (Symbol versus atom)
Note that in Definition 2.2.2, in the expression $\{a | a \in A\}$, the second a refers to the *symbol a*, while the first one refers to the *atom a*. This distinction should be clearly made, and it can be considered a tragic historical incident that these different notions have received the same notation.

Remark 2.2.4 (Range of a)
When we write "a" (or "b", or "c") in the context of an equality or a partial ordering, we mean this a to range over A_δ (provided, of course, deadlock is part of the relevant signature). When we write it in the context of a deduction rule, we mean it to range over A. In all other cases, or when we deviate from the above rule, we explicitly state whether it ranges over A or A_δ.

Definition 2.2.5 (Operator precedence)
Throughout this paper we adhere to the following operator precedence scheme, which consist of four categories of operators. The four categories, from strongly binding to weakly binding, are:

(i). all unary operators,

(ii). the sequential composition operator "$\cdot$",

(iii). all binary operators, except the "$+$" and the "$\cdot$",

(iv). the alternative composition operator "$+$".

Within one category, all operators bind equally strong.

Definition 2.2.6 (Axioms of BPA)
The process algebra BPA is axiomatized by Axioms A1–A5 shown in Table 1: BPA = A1–A5.

$$
\begin{array}{rcll}
x + y &=& y + x & \text{A1} \\
(x + y) + z &=& x + (y + z) & \text{A2} \\
x + x &=& x & \text{A3} \\
(x + y) \cdot z &=& x \cdot z + y \cdot z & \text{A4} \\
(x \cdot y) \cdot z &=& x \cdot (y \cdot z) & \text{A5}
\end{array}
$$

Table 1: Axioms of BPA.

Definition 2.2.7 (Notation Regarding Semantics I)
In order to define a semantics, we will use term deduction system semantics in the style of Section 2.2.3 of [13] (also called "Structured Operational Semantics" or "Plotkin-style semantics"). We use the notation $x \xrightarrow{a} x'$ to denote that x can do an a-step to x', $x \xrightarrow{a} \surd$ to denote that x can do an a-step and then terminate, $x \overset{a}{\nrightarrow}$ to denote that x cannot do an a-step, and $x \nrightarrow$ to denote that x cannot do any step at all.

For each process algebra we define, we will give a term deduction system. By using the concept of bisimulation (to be defined in Definition 2.2.10 on the following page), we then turn the term deduction system into a model of the given axioms.

Definition 2.2.8 (Semantics of BPA)
The semantics of BPA are given by the term deduction system $T(\text{BPA})$ induced by the deduction rules given in Table 2 and Table 3 on the following page.

$$a \xrightarrow{a} \surd$$

Table 2: Deduction rule for a.

$$\frac{x \xrightarrow{a} x'}{x+y \xrightarrow{a} x'} \qquad \frac{x \xrightarrow{a} \surd}{x+y \xrightarrow{a} \surd} \qquad \frac{x \xrightarrow{a} x'}{x \cdot y \xrightarrow{a} x' \cdot y}$$

$$\frac{y \xrightarrow{a} y'}{x+y \xrightarrow{a} y'} \qquad \frac{y \xrightarrow{a} \surd}{x+y \xrightarrow{a} \surd} \qquad \frac{x \xrightarrow{a} \surd}{x \cdot y \xrightarrow{a} y}$$

Table 3: Deduction rules for $+$ and $\cdot$.

Definition 2.2.9 (Symmetric Closure)
For a binary relation R, we denote its *symmetric closure* by R^S:

$$R^S = R \cup \{(y,x) \,|\, (x,y) \in R\}$$

Definition 2.2.10 (Bisimulation for BPA)
Bisimulation for BPA is defined as follows; a binary relation R on closed BPA terms is a bisimulation if the following transfer conditions hold for all closed BPA terms p and q:

(i). If $R^S(p,q)$ and $T(\mathrm{BPA}) \vDash p \xrightarrow{a} p'$, where $a \in A$, then there exists a closed term q', such that $T(\mathrm{BPA}) \vDash q \xrightarrow{a} q'$ and $R^S(p',q')$,

(ii). If $R^S(p,q)$ and $T(\mathrm{BPA}) \vDash p \xrightarrow{a} \surd$, where $a \in A$, then $T(\mathrm{BPA}) \vDash q \xrightarrow{a} \surd$.

Two BPA terms p and q are called *bisimilar*, notation $p \sim_{\mathrm{BPA}} q$, if there exists a bisimulation relation R such that $R(p,q)$.

Definition 2.2.11 (Bisimulation Model for BPA)
Using bisimulation, we can now construct a model of the axioms of BPA. In order to do this, we first need to know that bisimulation is a congruence with respect to all operators. In [24] it is proven that a sufficient condition for this is that the deduction rules satisfy the so called *panth* format. It is easy to check that this is indeed the case.

We then construct the bisimulation model for BPA by taking the equivalence classes of the set of all closed BPA terms with respect to bisimulation equivalence. As bisimulation is a congruence, the operators can be trivially defined on the equivalence classes. For example for the $+$ operator:

$$[x]_{\sim_{\mathrm{BPA}}} + [y]_{\sim_{\mathrm{BPA}}} = [x+y]_{\sim_{\mathrm{BPA}}}$$

Here $[x]_{\sim_{\mathrm{BPA}}}$ denotes the equivalence class of x with respect to the equivalence relation $\sim_{\mathrm{BPA}}$. The other operators are defined similarly.

Definition 2.2.12 (Basic Terms of BPA)
We define *basic terms* inductively as follows:

(i). Every $a \in A$ is a basic term,

(ii). if $a \in A$ and t is a basic term, then $a \cdot t$ is a basic term,

(iii). if t and s are basic terms, then $t + s$ is a basic term.

Definition 2.2.13 (Number of Symbols of a BPA Term)
We define $n(x)$, the number of symbols of x, inductively as follows:

(i). For $a \in A$, we define $n(a) = 1$,

(ii). for closed BPA terms x and y, we define $n(x + y) = n(x \cdot y) = n(x) + n(y) + 1$.

Definition 2.2.14 (Closed Terms of BPA)
We denote the set of all *closed terms* (i.e. terms not containing free variables) of BPA by $C(\mathrm{BPA})$. This notation extends as expected to other theories.

Remark 2.2.15 (Results from [13])
In [13] several results are given about elimination, soundness, and completeness with respect to BPA and BPA with some extensions. We will not repeat the proofs given for those results here, but instead refer to that article.

One might object that the proofs of [13] are often very sketchy ("Easy by induction."), and sometimes even incorrect (see for example Remark 2.4.18 on page 22 of our paper). Nevertheless, we felt that no useful purpose would be served by writing out those sketchy proofs in full here, as in a sense they are encompassed by the proofs regarding $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID (to be treated in Section 2.5), which we *do* give in full.

Proposition 2.2.16 (Elimination for BPA)
Let t be a closed BPA term. Then there is a basic term s such that $\mathrm{BPA} \vdash t = s$.

Proof This is Proposition 2.2.5 of [13]. See the proof given there. ■

Theorem 2.2.17 (Soundness of BPA)
The set of closed BPA terms modulo bisimulation equivalence is a model of BPA.

Proof This is Theorem 2.2.33 of [13]. See the proof given there. ■

Remark 2.2.18 (Proving Completeness using the Direct Method)
All completeness proofs regarding basic process algebras (i.e., all completeness proofs in Section 2) follow the same scheme, which we will outline in this remark in some detail, so we do not have to go over these details again and again in the actual proofs.

To prove completeness of process theory P, we first derive an auxiliary lemma "Towards Completeness of P" (see for example Lemma 2.2.19 on the following page) that contains sublemmata of the general form:

$$T(P) \models \ldots \implies P \vdash \ldots$$

Typically, each sublemma relates a certain transition in the term deduction system of P with a certain equality in P (some sublemmata slightly deviate from this format).

Armed with the implications proven in the "Towards..." lemma, we then set out to actually prove completeness (see for example Theorem 2.2.22 on page 11). This is done, using Lemma 2.2.20 on the following page, by proving that for all basic terms x and y of P we have that:

$$x + y \sim_P y \implies P \vdash x + y = y.$$

This part of the proof is done by induction on the number of symbols in x, using case distinction on the form of (basic term) x. The "Towards..." sublemmata are chosen in such a way that each case we encounter in completing our proof is now easily handled.

The proof method outlined in this remark is taken from [13].

Lemma 2.2.19 (Towards Completeness of BPA)
Let x and y be closed BPA terms. Then we have:

(i). $T(BPA) \models x \xrightarrow{a} \surd \implies BPA \vdash x = a + x$,

(ii). $T(BPA) \models x \xrightarrow{a} y \implies BPA \vdash x = a \cdot y + x$,

(iii). $T(BPA) \models x \xrightarrow{a} y \implies n(x) > n(y)$.

Proof For part (i) and (ii) we assume, by Proposition 2.2.16 and Theorem 2.2.17, without loss of generality, that x is a basic term, and then apply induction on the structure of basic terms. For part (iii) this does not work, as bisimulation obviously is not a congruence for $n(x)$. Therefore, in proving (iii) we use induction on the general structure of terms.

(i). Case 1: x is an atomic action. Because $T(\mathrm{BPA}) \models x \xrightarrow{a} \surd$, it must then be the case that $x \equiv a$. So we have $\mathrm{BPA} \vdash x = a = a+a = a+x$. Case 2: x is of the form atomic action followed by another basic term. This is in contradiction with $T(\mathrm{BPA}) \models x \xrightarrow{a} \surd$, so this case does not occur. Case 3: x is of the form $s+t$, where s and t are again basic terms. As $T(\mathrm{BPA}) \models s + t \xrightarrow{a} \surd$, necessarily $T(\mathrm{BPA}) \models s \xrightarrow{a} \surd$ or $T(\mathrm{BPA}) \models t \xrightarrow{a} \surd$. Therefore, by the induction hypothesis, $\mathrm{BPA} \vdash s = a + s$ or $\mathrm{BPA} \vdash t = a + t$. But then in both cases $\mathrm{BPA} \vdash x = s + t = a + s + t = a + x$.

(ii). Case 1: x is an atomic action. This is in contradiction with $T(\mathrm{BPA}) \models x \xrightarrow{a} y$, so this case does not occur. Case 2: x is of the form atomic action followed by another basic term. Then, because $T(\mathrm{BPA}) \models x \xrightarrow{a} y$, it must be that $x \equiv a \cdot y$. So, $\mathrm{BPA} \vdash x = a \cdot y = a \cdot y + a \cdot y = a \cdot y + x$. Case 3: x is of the form $s+t$, where s and t are again basic terms. As $T(\mathrm{BPA}) \models s+t \xrightarrow{a} y$, necessarily $T(\mathrm{BPA}) \models s \xrightarrow{a} y$ or $T(\mathrm{BPA}) \models t \xrightarrow{a} y$. Therefore, by the induction hypothesis, $\mathrm{BPA} \vdash s = a \cdot y + s$ or $\mathrm{BPA} \vdash t = a \cdot y + t$. So in both cases $\mathrm{BPA} \vdash x = s + t = a \cdot y + s + t = a \cdot y + x$.

(iii). Case 1: x is an atomic action. This is in contradiction with $T(\mathrm{BPA}) \models x \xrightarrow{a} y$, so this case does not occur. Case 2: x is of the form $s \cdot t$, for certain terms s and t. Then, by $T(\mathrm{BPA}) \models x \xrightarrow{a} y$, we either have $T(\mathrm{BPA}) \models s \xrightarrow{a} \surd$ and $y \equiv t$, or we have $T(\mathrm{BPA}) \models s \xrightarrow{a} s'$ and $y \equiv s' \cdot t$ for some term s'. In the first case, we have $n(x) = n(s \cdot t) = n(s) + n(t) + 1 > n(t) = n(y)$, and in the second we can apply the induction hypothesis to arrive at $n(s) > n(s')$, so we get $n(x) = n(s \cdot t) = n(s) + n(t) + 1 > n(s') + n(t) + 1 = n(s' \cdot t) = n(y)$. Case 3: x is of the form $s+t$, for certain terms s and t. As $T(\mathrm{BPA}) \models s+t \xrightarrow{a} y$, necessarily $T(\mathrm{BPA}) \models s \xrightarrow{a} y$ or $T(\mathrm{BPA}) \models t \xrightarrow{a} y$. Therefore, by the induction hypothesis, $n(s) > n(y)$ or $n(t) > n(y)$. As n ranges over the positive naturals only, in both cases $n(x) = n(s + t) = n(s) + n(t) + 1 > n(y)$.

■

Lemma 2.2.20 (Towards Completeness of BPA)
In order to prove for all BPA terms x and y that:

$$x \sim_{BPA} y \implies BPA \vdash x = y \qquad (*)$$

it is sufficient to prove that:

$$x + y \sim_{BPA} y \implies BPA \vdash x + y = y. \qquad (\dagger)$$

Proof Assume (†) and the left-hand side of ($*$) to hold. Then prove the right-hand side of ($*$). This is done as follows: by Theorem 2.2.17, the fact that $\sim_{\text{BPA}}$ is a congruence, and Axiom A3, for all bisimilar BPA terms x and y we have $x + y \sim_{\text{BPA}} y + y \sim_{\text{BPA}} y$ and $y + x \sim_{\text{BPA}} x + x \sim_{\text{BPA}} x$. Therefore, by (†), also $\text{BPA} \vdash x + y = y$ and $\text{BPA} \vdash y + x = x$. But that gives us: $\text{BPA} \vdash x = y + x = x + y = y$. ■

Corollary 2.2.21 (Generalization of Lemma 2.2.20)
Lemma 2.2.20 generalizes from BPA to any equational process theory P with corresponding term deduction system $T(P)$, provided:

(i). P is a sound axiomatization of $T(P)$,

(ii). $\sim_P$ is a congruence for the function symbols from the signature of P, and,

(iii). P contains the axioms of BPA.

Proof As the proof of Lemma 2.2.20 only depends on the soundness of BPA with respect to $T(\text{BPA})$, the fact that $\sim_P$ is a congruence for P, and the axioms of BPA, the proof is trivially valid for P too. ■

Theorem 2.2.22 (Completeness of BPA)
The axiom system BPA is a complete axiomatization of the set of closed BPA terms modulo bisimulation equivalence.

Proof Let x and y be bisimilar closed BPA terms. We have to prove that $\text{BPA} \vdash x = y$. With the aid of Proposition 2.2.16 and Theorem 2.2.17, it is enough to prove this for basic terms. By Lemma 2.2.20 it is even enough to prove for all basic terms x and y that:

$$x + y \sim_{\text{BPA}} y \implies \text{BPA} \vdash x + y = y.$$

We will prove this by induction on $n(x)$, using 2.2.19(iii) and case distinction on the form of basic term x. Case 1: x is of the form a, for $a \in A$. Then $T(\text{BPA}) \vDash x \xrightarrow{a} \surd$, so $T(\text{BPA}) \vDash x + y \xrightarrow{a} \surd$, and because $x + y \sim_{\text{BPA}} y$ we have $T(\text{BPA}) \vDash y \xrightarrow{a} \surd$, so with Lemma 2.2.19(i) we find that $\text{BPA} \vdash x + y = y$. This proves the basis of our induction. Case 2: x is of the form $a \cdot s$, where $a \in A$, and s again a basic BPA term. Then $T(\text{BPA}) \vDash x \xrightarrow{a} s$, and therefore $T(\text{BPA}) \vDash x + y \xrightarrow{a} s$, so because $x + y \sim_{\text{BPA}} y$ there is an s' with $T(\text{BPA}) \vDash y \xrightarrow{a} s'$ and $s \sim_{\text{BPA}} s'$. But then by Theorem 2.2.17 and Axiom A3 also $s + s' \sim_{\text{BPA}} s'$ and $s' + s \sim_{\text{BPA}} s$ and with induction (note that $n(s) < n(x)$) we find $\text{BPA} \vdash s + s' = s'$ and $\text{BPA} \vdash s' + s = s$. So $\text{BPA} \vdash s = s'$. Now $\text{BPA} \vdash x + y = a \cdot s + y = a \cdot s' + y = y$ with Lemma 2.2.19(ii). Case 3: x is of the form $s + t$, for certain basic BPA terms s and t. Since $x + y \sim_{\text{BPA}} y$, we also have $s + y \sim_{\text{BPA}} y$ and $t + y \sim_{\text{BPA}} y$. By induction $\text{BPA} \vdash s + y = y$ and $\text{BPA} \vdash t + y = y$. So $\text{BPA} \vdash x + y = s + t + y = s + y = y$. ■

2.3 Soundness and Completeness of BPA$_\delta$

Definition 2.3.1 (Signature of BPA$_\delta$)
The signature of BPA$_\delta$ consists of the *atomic actions* $\{a | a \in A\}$, the *deadlock constant* δ, the *alternative composition operator* $+$, and the *sequential composition operator* $\cdot$.

$$x + \delta = x \qquad \text{A6}$$
$$\delta \cdot x = \delta \qquad \text{A7}$$

Table 4: Axioms for δ.

Definition 2.3.2 (Axioms of BPA$_\delta$)
The process algebra BPA$_\delta$ is axiomatized by the axioms of BPA given in Definition 2.2.6 on page 7 and Axioms A6–A7 shown in Table 4: BPA$_\delta$ = A1–A7.

Remark 2.3.3 (Axiom A6 versus Axiom A6A)
Note that in the presence of the other axioms of BPA, Axiom A6 given in Table 4 is equivalent, for *closed* BPA$_\delta$ terms, with Axiom A6A given in Table 5. Therefore we could replace A6 in BPA by A6A without affecting the soundness or completeness of the resulting theory.

One such reason to do so, could be the fact that A6A remains valid in all discrete-time process algebras we will describe, whereas A6 does not. Still, for historical reasons, we prefer A6 to be used in the definition of BPA$_\delta$. We will later return to this subject in Remark 2.5.3 on page 22 and Remark 2.7.3 on page 45.

$$a + \delta = a \qquad \text{A6A}$$

Table 5: Alternative for Axiom A6.

Definition 2.3.4 (Semantics of BPA$_\delta$)
The semantics of BPA$_\delta$ are given by the term deduction system $T(\text{BPA}_\delta)$ induced by the deduction rules given in Table 2 on page 7 and Table 3 on page 8.

Note that this term deduction system $T(\text{BPA}_\delta)$ is practically identical to the term deduction system $T(\text{BPA})$ given in Definition 2.2.8 on page 7, as there are no deduction rules for δ. However, $T(\text{BPA}_\delta)$ does differ from $T(\text{BPA})$ in the fact that it contains the symbol δ in its signature.

Definition 2.3.5 (Bisimulation and Bisimulation Model for BPA$_\delta$)
Bisimulation for BPA$_\delta$ and the corresponding bisimulation model are defined in the same way as for BPA. Replace "BPA" by "BPA$_\delta$" in Definition 2.2.10 on page 8 and Definition 2.2.11 on page 8.

Definition 2.3.6 (Basic Terms of BPA$_\delta$)
We define *δ-basic terms* inductively as follows:

(i). Every $a \in A_\delta$ is a δ-basic term, (ii). if $a \in A_\delta$ and t is a δ-basic term, then $a \cdot t$ is a δ-basic term,

(iii). if t and s are δ-basic terms, then $t + s$ is a δ-basic term.

From now on, if we speak of basic terms in the context of BPA_δ, we mean δ-basic terms.

Remark 2.3.7 (Definition of Basic Terms)
Usually the basic terms of BPA_δ are defined a bit differently with respect to deadlock: $\delta \cdot t$ for some basic term t is usually not considered basic. We chose to deviate from established practice because it made our proofs quite a bit shorter. The reason is that in this way we get rid of a nasty case distinction that would otherwise have popped up in just about every other line of our proofs.

Definition 2.3.8 (Number of Symbols of a BPA_δ Term)
We define $n(x)$, the number of symbols of x, inductively as follows:

(i). For $a \in A_\delta$, we define $n(a) = 1$,

(ii). for closed BPA_δ terms x and y, we define $n(x + y) = n(x \cdot y) = n(x) + n(y) + 1$.

Proposition 2.3.9 (Elimination for BPA_δ)
Let t be a closed BPA_δ term. Then there is a basic term s such that $\mathrm{BPA}_\delta \vdash t = s$.

Proof This is Proposition 2.5.3 of [13], see the proof given there. Note that although [13] uses a slightly different definition of basic terms, their definition is narrower, so their elimination result also holds for our definition of basic terms. ■

Theorem 2.3.10 (Soundness of BPA_δ)
The set of closed BPA_δ terms modulo bisimulation equivalence is a model of BPA_δ.

Proof This is Theorem 2.5.4 of [13]. See the proof given there. ■

Theorem 2.3.11 (Completeness of BPA_δ)
The axiom system BPA_δ is a complete axiomatization of the set of closed BPA_δ terms modulo bisimulation equivalence.

Proof Since there are no transitions for the new constant δ, this is proven in the same way as Theorem 2.2.22. ■

2.4 Soundness and Completeness of $\mathrm{BPA}^-_{\mathrm{drt}}$–$\delta$

Definition 2.4.1 (Signature of $\mathrm{BPA}^-_{\mathrm{drt}}$–$\delta$)
The signature of $\mathrm{BPA}^-_{\mathrm{drt}}$–$\delta$ consists of the *undelayable atomic actions* $\{\underline{\underline{a}} | a \in A\}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, and the *time unit delay operator* σ_{rel}.

Definition 2.4.2 (Axioms of $\mathrm{BPA}^-_{\mathrm{drt}}$–$\delta$)
The process algebra $\mathrm{BPA}^-_{\mathrm{drt}}$–$\delta$ is axiomatized by the axioms of BPA given in Definition 2.2.6 on page 7 and Axioms DRT1–DRT2 shown in Table 6 on the next page: $\mathrm{BPA}^-_{\mathrm{drt}}$–$\delta$ = A1–A5 + DRT1–DRT2.

$$\sigma_{\mathrm{rel}}(x) + \sigma_{\mathrm{rel}}(y) = \sigma_{\mathrm{rel}}(x + y) \qquad \text{DRT1}$$
$$\sigma_{\mathrm{rel}}(x) \cdot y = \sigma_{\mathrm{rel}}(x \cdot y) \qquad \text{DRT2}$$

Table 6: Axioms for σ_{rel}.

Definition 2.4.3 (Notation Regarding Semantics II)
Next to the deduction rule notations $x \xrightarrow{a} x'$, $x \xrightarrow{a} \surd$, and $x \stackrel{a}{\nrightarrow}$, and $x \nrightarrow$ introduced in Definition 2.2.7 on page 7, we now also use $x \xrightarrow{\sigma} x'$ to denote that x can do a σ-step to x' (i.e., move to the following time-slice and become x'), and $x \stackrel{\sigma}{\nrightarrow}$ to denote that x cannot do an σ-step.

Definition 2.4.4 (Semantics of BPA$^{-}_{\mathrm{drt}}$–δ)
The semantics of BPA$^{-}_{\mathrm{drt}}$–δ are given by the term deduction system $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta)$ induced by the deduction rules given in Table 3 on page 8 and Table 7.

$$\underline{\underline{a}} \xrightarrow{a} \surd \qquad \sigma_{\mathrm{rel}}(x) \xrightarrow{\sigma} x \qquad \frac{x \xrightarrow{\sigma} x'}{x \cdot y \xrightarrow{\sigma} x' \cdot y}$$

$$\frac{x \xrightarrow{\sigma} x',\ y \xrightarrow{\sigma} y'}{x + y \xrightarrow{\sigma} x' + y'} \qquad \frac{x \xrightarrow{\sigma} x',\ y \stackrel{\sigma}{\nrightarrow}}{x + y \xrightarrow{\sigma} x'} \qquad \frac{x \stackrel{\sigma}{\nrightarrow},\ y \xrightarrow{\sigma} y'}{x + y \xrightarrow{\sigma} y'}$$

Table 7: Deduction rules for $\underline{\underline{a}}$ and σ_{rel}.

Definition 2.4.5 (Bisimulation for BPA$^{-}_{\mathrm{drt}}$–δ)
Bisimulation for BPA$^{-}_{\mathrm{drt}}$–δ is defined as follows; a binary relation R on closed BPA$^{-}_{\mathrm{drt}}$–δ terms is a bisimulation if the following transfer conditions hold for all closed BPA$^{-}_{\mathrm{drt}}$–δ terms p and q:

(i). If $R^{\mathrm{S}}(p,q)$ and $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models p \xrightarrow{a} p'$, where $a \in A$, then there exists a closed term q', such that $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models q \xrightarrow{a} q'$ and $R^{\mathrm{S}}(p',q')$,

(ii). If $R^{\mathrm{S}}(p,q)$ and $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models p \xrightarrow{\sigma} p'$, then there exists a closed term q', such that $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models q \xrightarrow{\sigma} q'$ and $R^{\mathrm{S}}(p',q')$,

(iii). If $R^{\mathrm{S}}(p,q)$ and $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models p \xrightarrow{a} \surd$, where $a \in A$, then $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models q \xrightarrow{a} \surd$.

Two BPA$^{-}_{\mathrm{drt}}$–δ terms p and q are bisimilar, notation $p \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta} q$, if there exists a bisimulation relation R such that $R(p,q)$.

Definition 2.4.6 (Bisimulation Model for BPA$^{-}_{\mathrm{drt}}$–δ)
The bisimulation model for BPA$^{-}_{\mathrm{drt}}$–δ is defined in the same way as for BPA. Replace "BPA" by "BPA$^{-}_{\mathrm{drt}}$–δ" in Definition 2.2.11 on page 8.

Definition 2.4.7 (Basic Terms of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$)
We define *σ-basic terms* inductively as follows:

(i). For every $a \in A$, the atomic action $\underline{\underline{a}}$ is a σ-basic term,

(ii). if $a \in A$ and t is a σ-basic term, then $\underline{\underline{a}} \cdot t$ is a σ-basic term,

(iii). if t and s are σ-basic terms, then $t + s$ is a σ-basic term,

(iv). if t is a basic term, then $\sigma_{\mathrm{rel}}(t)$ is a σ-basic term.

From now on, if we speak of basic terms in the context of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$, we mean σ-basic terms.

Definition 2.4.8 (Number of Symbols of a $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ Term)
We define $n(x)$, the number of symbols of x, inductively as follows:

(i). For $a \in A$, we define $n(\underline{\underline{a}}) = 1$,

(ii). for closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ terms x and y, we define $n(x+y) = n(x \cdot y) = n(x) + n(y) + 1$,

(iii). for a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ term x, we define $n(\sigma_{\mathrm{rel}}(x)) = n(x) + 1$.

Definition 2.4.9 (Symbol for a Chain of σ's)
We will write $x \overset{\sigma}{\Longrightarrow} y$ to indicate that x can reach y by doing zero or more σ-transitions. Formally: $\overset{\sigma}{\Longrightarrow}$ denotes the transitive, reflexive closure of $\overset{\sigma}{\to}$.

Remark 2.4.10 (Proving Elimination using Term Rewriting Analysis)
To prove elimination we will use, where possible, a method that is based on term rewriting analysis. This method works by associating a term rewriting system to an equational specification, and then proving that this term rewriting system is strongly normalizing and its normal forms are basic terms. See Theorem 2.4.11 for an example of this method.

Theorem 2.4.11 (Elimination for $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$)
Let t be a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ term. Then there is a basic term s such that $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta \vdash s = t$.

Proof This theorem is proven as follows. First a number of axioms of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ are selected, and subsequently oriented as rewriting rules. This gives us a term rewriting system. Then it is proven that this term rewriting system is strongly normalizing and that every normal form of a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ term is a basic term. In this way a recipe is obtained for transforming a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ term into a basic term.

The rewriting rules of the term rewriting system for $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ are given in Table 8 on the next page. The proof that this term rewriting system is strongly normalizing uses the method of the lexicographical path ordering.

The well-founded ordering $>$ on constants and function symbols is the following:

$$\cdot > + > \sigma_{\mathrm{rel}} > \underline{\underline{a}}$$

To $\cdot$ we assign the lexicographical status for the first argument. Now we show that the left-hand side of every rewriting rule is bigger than the right-hand side with respect to the ordering $\succ_{\mathrm{lpo}}$. This is done by the following reductions (taken from [13]):

$$\begin{aligned}(x+y) \cdot z &\succ_{\mathrm{lpo}} (x+y) \cdot^{\star} z \succ_{\mathrm{lpo}} (x+y) \cdot^{\star} z + (x+y) \cdot^{\star} z \succ_{\mathrm{lpo}} (x +^{\star} y) \cdot z + (x +^{\star} y) \cdot z \\ &\succ_{\mathrm{lpo}} x \cdot z + y \cdot z\end{aligned}$$

$$(x + y) \cdot z \to x \cdot z + y \cdot z \qquad \text{RA4}$$
$$(x \cdot y) \cdot z \to x \cdot (y \cdot z) \qquad \text{RA5}$$
$$\sigma_{\text{rel}}(x) \cdot y \to \sigma_{\text{rel}}(x \cdot y) \qquad \text{RDRT2}$$

Table 8: Term Rewriting System for $\text{BPA}^{-}_{\text{drt}}$–$\delta$.

$$\begin{aligned}(x \cdot y) \cdot z &\succ_{\text{lpo}} (x \cdot y) \cdot^{\star} z \succ_{\text{lpo}} (x \cdot^{\star} y) \cdot ((x \cdot y) \cdot^{\star} z) \succ_{\text{lpo}} x \cdot ((x \cdot^{\star} y) \cdot z) \\ &\succ_{\text{lpo}} x \cdot (y \cdot z) \\ \sigma_{\text{rel}}(x) \cdot y &\succ_{\text{lpo}} \sigma_{\text{rel}}(x) \cdot^{\star} y \succ_{\text{lpo}} \sigma_{\text{rel}}(\sigma_{\text{rel}}(x) \cdot^{\star} y) \succ_{\text{lpo}} \sigma_{\text{rel}}(\sigma^{\star}_{\text{rel}}(x) \cdot y) \\ &\succ_{\text{lpo}} \sigma_{\text{rel}}(x \cdot y)\end{aligned}$$

Next, we will prove that the normal forms of the closed $\text{BPA}^{-}_{\text{drt}}$–$\delta$ terms are basic terms. Thereto, suppose that s is a normal form of some closed $\text{BPA}^{-}_{\text{drt}}$–$\delta$ term. Furthermore, suppose that s is not a basic term. Let s' denote the smallest subterm of s which is not a basic term. Note that, consequently, all proper subterms of s' are basic terms. Then we can prove that s' is not a normal form by case analysis. We distinguish all possible cases:

(i). s' is an atomic action. But then s' is a basic term. This is in contradiction with the assumption that s' is not a basic term, so this case does not occur.

(ii). s' is of the form $s'_1 \cdot s'_2$ for basic terms s'_1 and s'_2. With case analysis on the structure of basic term s'_1:

- (a) If s'_1 is an atomic action $\underline{\underline{a}}$ then $s'_1 \cdot s'_2$ is a basic term, and so s' is a basic term which again contradicts the assumption that s' is not a basic term. This case can therefore not occur.
- (b) If s'_1 is of the form $\underline{\underline{a}} \cdot t$ for some atomic action $\underline{\underline{a}}$ and basic term t, then rewriting rule RA5 can be applied. So, s' is not a normal form.
- (c) If s'_1 is of the form $t_1 + t_2$ for t_1 and t_2 basic terms. Then rewriting rule RA4 is applicable. Therefore, s' is not a normal form.
- (d) If s'_1 is of the form $\sigma_{\text{rel}}(t)$ for some basic term t. Then rewriting rule RDRT2 is applicable. So, s' is not a normal form.

(iii). s' is of the form $s'_1 + s'_2$ for basic terms s'_1 and s'_2. In this case s' would be a basic term, which contradicts the assumption that s' is not a basic term. Therefore, this case cannot occur.

(iv). s' is of the form $\sigma_{\text{rel}}(s'')$ for some basic term s''. But then s' is basic term too, so the case does not occur.

In any case that can occur it follows that s' is not a normal form. Since s' is a subterm of s, we conclude that s is not a normal form. This contradicts the assumption that s is a normal form. From this contradiction we conclude that s is a basic term, which completes the proof. ■

Remark 2.4.12 (Elimination for BPA$^{-}_{\mathrm{drt}}$-δ)
Elimination for BPA$^{-}_{\mathrm{drt}}$-δ is also claimed (without proof) in Theorem 2.12.3 of [13] (where BPA$^{-}_{\mathrm{drt}}$-δ is called BPA$_{\mathrm{dt}}$).

Remark 2.4.13 (Proving Soundness using the Direct Method)
Most of the soundness proofs given in this paper follow the same scheme, which we will outline in this remark in some detail, so we do not have to go over these details again and again in the actual proofs.

To prove the soundness of a certain axiom of the general form

$$t_l(x_1, \ldots, x_n) = t_r(x_1, \ldots, x_n)$$

where t_l and t_r are process expressions in the free variables $x_1, \ldots, x_n$ for some $n \geq 0$, with respect to some bisimulation model, we proceed as follows. First, we give a relation R, which will be a binary relation on closed terms. Then, we show that this R is a bisimulation relation that for all closed instantiations of $x_1, \ldots, x_n$ relates the left-hand and right-hand side of the axiom. This involves two steps:

(i). R should relate both sides of the axiom for all closed terms, i.e., for all closed instantiations of $x_1, \ldots, x_n$ we should have that

$$(t_l(x_1, \ldots, x_n), t_r(x_1, \ldots, x_n)) \in R.$$

This is mostly so trivial that we do not mention it at all.

(ii). R should be a bisimulation. In order to prove that, we show that for all closed terms s, t in the relation the transfer conditions from the definition of bisimulation are satisfied.

For example, in proving soundness of an axiom of BPA$^{-}_{\mathrm{drt}}$-δ, we have to show that for all closed term s, t such that $(s, t) \in R$, we have that for any transition $s \xrightarrow{u} s'$ (where $u \in A_\sigma$), there is a corresponding transition $t \xrightarrow{u} t'$ such that $(s', t') \in R$, and vice versa, for any transition $t \xrightarrow{u} t'$, there is a corresponding transition $s \xrightarrow{u} s'$ such that again $(s', t') \in R$. This part of the proof is done using case distinction on the different kinds of steps that are possible (an action, a time step, termination).

Note that the "vice versa" part of proof obligation (ii) results from the fact that the transfer conditions for bisimulation (see for example Definition 2.4.5 on page 14) are defined with respect to the symmetric closure of R.

This completes the general outline of our soundness proofs. In the actual proofs we will sometimes slightly deviate from it, for example because we re-use an earlier proof, or because we do not faithfully list the too trivial proof obligations. The above described general outline will however remain visible in the background. Finally, note that in soundness proofs we do not explicitly indicate the term reduction system in which a transition occurs; so instead of $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models x \xrightarrow{a} y$, we simply write $x \xrightarrow{a} y$. This is no problem, as the theory in which we are working is always clear.

The proof method outlined in this remark is taken from [13].

Theorem 2.4.14 (Soundness of BPA$^{-}_{\mathrm{drt}}$-δ)
The set of closed BPA$^{-}_{drt}$-δ terms modulo bisimulation equivalence is a model of BPA$^{-}_{drt}$-δ.

Proof Since bisimulation equivalence is a congruence (see for example Theorem 2.2.32 of [13]), also for the new operators, we only need to verify the soundness of every closed instantiation of the axioms. We do this by giving a relation R for every axiom and we prove that this relation is a bisimulation relation for every closed instantiation of the left-hand and right-hand sides of the axiom.

In the setting of BPA soundness of Axioms A1–A5 has already been proven (see for example Theorem 2.2.33 of [13]).

The theory $\mathrm{BPA}^{-}_{\mathrm{drt}}$-$\delta$ adds to this the possibility to perform σ-transitions. However, any term headed by the operator added to BPA is not capable of performing a transition labeled by an atomic action. So, we argue that the left-hand side and the right-hand side of Axioms A1–A5 can perform exactly the same transitions labeled by an atomic action in the theory $\mathrm{BPA}^{-}_{\mathrm{drt}}$-$\delta$ as in the theory BPA. Therefore, we only consider the transitions labeled by σ for Axioms A1–A5. For Axioms DRT1–DRT2 we, of course, have to consider both the transitions labeled by an atomic action and the transitions labeled by a σ.

Finally, note that $\mathrm{BPA}^{-}_{\mathrm{drt}}$-$\delta$ contains atomic actions of the form $\underline{\underline{a}}$ whereas BPA had actions of the form a. This is however not relevant for the purpose of extending the soundness proofs of Axioms A1–A5 from BPA to $\mathrm{BPA}^{-}_{\mathrm{drt}}$-$\delta$, as neither a nor $\underline{\underline{a}}$ appears in Axioms A1–A5, and both have exactly the same deduction rules (see Table 2 on page 7 and Table 7 on page 14).

Axiom A1 Take the relation:

$$R = \{(s,s), (s+t, t+s) \mid s,t \in C(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta)\}$$

First we look at the transitions of the left-hand side. Suppose $s + t \xrightarrow{\sigma} p$. Then one of the following situations occurs:

(i). $s \xrightarrow{\sigma} p$ and $t \not\xrightarrow{\sigma}$: then also $t + s \xrightarrow{\sigma} p$, and note that $(p,p) \in R$.

(ii). $s \not\xrightarrow{\sigma}$ and $t \xrightarrow{\sigma} p$: then also $t + s \xrightarrow{\sigma} p$, and note that $(p,p) \in R$.

(iii). $s \xrightarrow{\sigma} p_1$ and $t \xrightarrow{\sigma} p_2$ and $p \equiv p_1 + p_2$: then $t + s \xrightarrow{\sigma} p_2 + p_1$, and note that $(p_1 + p_2, p_2 + p_1) \in R$.

The proof for the right-hand side is analogous.

Axiom A2 Take the relation:

$$R = \{(s,s), ((s+t)+u, s+(t+u)) \mid s,t,u \in C(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta)\}$$

First we look at the transitions of the left-hand side. Suppose $(s+t)+u \xrightarrow{\sigma} p$. Then one of the following situations occurs:

(i). $s + t \xrightarrow{\sigma} p$ and $u \not\xrightarrow{\sigma}$: then the transition $s + t \xrightarrow{\sigma} p$ must be due to one of the following:

- (a) $s \xrightarrow{\sigma} p$ and $t \not\xrightarrow{\sigma}$: in that case $s \xrightarrow{\sigma} p$ and $t + u \not\xrightarrow{\sigma}$. So, $s + (t+u) \xrightarrow{\sigma} p$, and note that $(p,p) \in R$.
- (b) $s \not\xrightarrow{\sigma}$ and $t \xrightarrow{\sigma} p$: in that case $s \not\xrightarrow{\sigma}$ and $t + u \xrightarrow{\sigma} p$. Therefore, $s + (t+u) \xrightarrow{\sigma} p$.
- (c) $s \xrightarrow{\sigma} p_1$ and $t \xrightarrow{\sigma} p_2$ and $p \equiv p_1 + p_2$: in that case $s \xrightarrow{\sigma} p_1$ and $t + u \xrightarrow{\sigma} p_2$. Therefore, $s + (t+u) \xrightarrow{\sigma} p_1 + p_2$, and note that $(p,p) \in R$.

(ii). $s + t \overset{\sigma}{\nrightarrow}$ and $u \xrightarrow{\sigma} p$: then $s \overset{\sigma}{\nrightarrow}$, $t \overset{\sigma}{\nrightarrow}$, and $u \xrightarrow{\sigma} p$. So, $s \overset{\sigma}{\nrightarrow}$ and $t + u \xrightarrow{\sigma} p$. Therefore, $s + (t + u) \xrightarrow{\sigma} p$, and note that $(p, p) \in R$.

(iii). $s + t \xrightarrow{\sigma} p_1$ and $u \xrightarrow{\sigma} p_2$ and $p \equiv p_1 + p_2$: then the transition $s + t \xrightarrow{\sigma} p_1$ must be due to one of the following:

(a) $s \xrightarrow{\sigma} p_1$ and $t \overset{\sigma}{\nrightarrow}$: in that case $s \xrightarrow{\sigma} p_1$ and $t + u \xrightarrow{\sigma} p_2$. So, $s + (t + u) \xrightarrow{\sigma} p_1 + p_2$, and note that $(p, p) \in R$.

(b) $s \overset{\sigma}{\nrightarrow}$ and $t \xrightarrow{\sigma} p_1$: in that case $s \overset{\sigma}{\nrightarrow}$ and $t + u \xrightarrow{\sigma} p_1 + p_2$. Therefore, $s + (t + u) \xrightarrow{\sigma} p_1 + p_2$, and note that $(p, p) \in R$.

(c) $s \xrightarrow{\sigma} q_1$ and $t \xrightarrow{\sigma} q_2$ and $p_1 \equiv q_1 + q_2$: in that case $s \xrightarrow{\sigma} q_1$ and $t + u \xrightarrow{\sigma} q_2 + p_2$. Therefore, $s + (t + u) \xrightarrow{\sigma} q_1 + (q_2 + p_2)$, and note that $((q_1 + q_2) + p_2, q_1 + (q_2 + p_2)) \in R$.

The proof for the right-hand side is analogous.

Axiom A3 Take the relation:

$$R = \{(s, s), (s + s, s) \mid s \in C(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta)\}$$

First we look at the transitions of the left-hand side. Suppose $s + s \xrightarrow{\sigma} p$. Then $s \xrightarrow{\sigma} p'$ and $p \equiv p' + p'$. Then $s \xrightarrow{\sigma} p'$, and note that $(p' + p', p') \in R$.

The proof for the right-hand side is analogous.

Axiom A4 Take the relation:

$$R = \{(s, s), ((s + t) \cdot u, s \cdot u + t \cdot u) \mid s, t, u \in C(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta)\}$$

First we look at the transitions of the left-hand side. Suppose $(s + t) \cdot u \xrightarrow{\sigma} p$. Then $s + t \xrightarrow{\sigma} p'$ and $p \equiv p' \cdot u$. The transition $s + t \xrightarrow{\sigma} p'$ must be due to one of the following:

(i). $s \xrightarrow{\sigma} p'$ and $t \overset{\sigma}{\nrightarrow}$: then $s \cdot u + t \cdot u \xrightarrow{\sigma} p' \cdot u$, and note that $(p, p) \in R$.

(ii). $s \overset{\sigma}{\nrightarrow}$ and $t \xrightarrow{\sigma} p'$: analogous to the previous case.

(iii). $s \xrightarrow{\sigma} p_1$ and $t \xrightarrow{\sigma} p_2$ and $p' \equiv p_1 + p_2$: then $s \cdot u + t \cdot u \xrightarrow{\sigma} p_1 \cdot u + p_2 \cdot u$, and note that $((p_1 + p_2) \cdot u, p_1 \cdot u + p_2 \cdot u) \in R$.

Secondly, we look at the transitions of the right-hand side. Suppose $s \cdot u + t \cdot u \xrightarrow{\sigma} p$. This must be due to one of the following:

(i). $s \cdot u \xrightarrow{\sigma} p$ and $t \cdot u \overset{\sigma}{\nrightarrow}$: then $s \xrightarrow{\sigma} p'$ and $p \equiv p' \cdot u$. Also $t \overset{\sigma}{\nrightarrow}$. Therefore, $(s + t) \xrightarrow{\sigma} p'$ and $(s + t) \cdot u \xrightarrow{\sigma} p' \cdot u$, and note that $(p, p) \in R$.

(ii). $s \cdot u \overset{\sigma}{\nrightarrow}$ and $t \cdot u \xrightarrow{\sigma} p$: analogous to the previous case.

(iii). $s \cdot u \xrightarrow{\sigma} p_1$ and $t \cdot u \xrightarrow{\sigma} p_2$ and $p \equiv p_1 + p_2$: then $s \xrightarrow{\sigma} q_1$ and $p_1 \equiv q_1 \cdot u$ and $t \xrightarrow{\sigma} q_2$ and $p_2 \equiv q_2 \cdot u$. Therefore, $(s + t) \xrightarrow{\sigma} q_1 + q_2$ and $(s + t) \cdot u \xrightarrow{\sigma} (q_1 + q_2) \cdot u$, and note that $((q_1 + q_2) \cdot u, q_1 \cdot u + q_2 \cdot u) \in R$.

Axiom A5 Take the relation:

$$R = \{(s, s), ((s \cdot t) \cdot u, s \cdot (t \cdot u)) \mid s, t, u \in C(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta)\}$$

First we look at the transitions of the left-hand side. Suppose $(s \cdot t) \cdot u \xrightarrow{\sigma} p$. Then this must be due to $s \xrightarrow{\sigma} p'$ and $p \equiv (p' \cdot t) \cdot u$. So, $s \cdot (t \cdot u) \xrightarrow{\sigma} p' \cdot (t \cdot u)$, and note that $((p' \cdot t) \cdot u, p' \cdot (t \cdot u)) \in R$.

The proof for the right-hand side is analogous.

Axiom DRT1 Take the relation:

$$R = \{(s,s), (\sigma_{\mathrm{rel}}(s) + \sigma_{\mathrm{rel}}(t), \sigma_{\mathrm{rel}}(s+t)) \mid s,t \in C(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta)\}$$

We look at the transitions of both sides at the same time. First, note that $\sigma_{\mathrm{rel}}(s) + \sigma_{\mathrm{rel}}(t) \stackrel{a}{\nrightarrow}$ and $\sigma_{\mathrm{rel}}(s+t) \stackrel{a}{\nrightarrow}$. Secondly, $\sigma_{\mathrm{rel}}(s) + \sigma_{\mathrm{rel}}(t) \xrightarrow{\sigma} p$ iff $p \equiv s + t$ iff $\sigma_{\mathrm{rel}}(s + t) \xrightarrow{\sigma} p$, and note that $(p,p) \in R$.

Axiom DRT2 Take the relation:

$$R = \{(s,s), (\sigma_{\mathrm{rel}}(s) \cdot t, \sigma_{\mathrm{rel}}(s \cdot t)) \mid s,t \in C(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta)\}$$

We look at the transitions of both sides at the same time. First, note that $\sigma_{\mathrm{rel}}(s) \cdot t \stackrel{a}{\nrightarrow}$ and $\sigma_{\mathrm{rel}}(s \cdot t) \stackrel{a}{\nrightarrow}$. Secondly, $\sigma_{\mathrm{rel}}(s) \cdot t \xrightarrow{\sigma} p$ iff $p \equiv s \cdot t$ iff $\sigma_{\mathrm{rel}}(s \cdot t) \xrightarrow{\sigma} p$, and note that $(p,p) \in R$.

■

Remark 2.4.15 (Soundness of $\mathrm{BPA}^{-}_{\mathrm{drt}}$-$\delta$)
Soundness of $\mathrm{BPA}^{-}_{\mathrm{drt}}$-$\delta$ is also claimed (without proof) in Theorem 2.12.4 of [13] (where $\mathrm{BPA}^{-}_{\mathrm{drt}}$-$\delta$ is called $\mathrm{BPA}_{\mathrm{dt}}$).

Lemma 2.4.16 (Towards Completeness of $\mathrm{BPA}^{-}_{\mathrm{drt}}$-$\delta$)
Let x and y be closed BPA^{-}_{drt}-δ terms. Then we have:

(i). $T(BPA^{-}_{drt}\text{-}\delta) \models x \xrightarrow{a} \surd \implies BPA^{-}_{drt}\text{-}\delta \vdash x = \underline{\underline{a}} + x,$

(ii). $T(BPA^{-}_{drt}\text{-}\delta) \models x \xrightarrow{a} y \implies BPA^{-}_{drt}\text{-}\delta \vdash x = \underline{\underline{a}} \cdot y + x,$

(iii). $T(BPA^{-}_{drt}\text{-}\delta) \models x \xrightarrow{\sigma} y \implies BPA^{-}_{drt}\text{-}\delta \vdash x = \sigma_{rel}(y) + x,$

(iv). $T(BPA^{-}_{drt}\text{-}\delta) \models x \xrightarrow{a} y \implies n(x) > n(y),$

(v). $T(BPA^{-}_{drt}\text{-}\delta) \models x \xrightarrow{\sigma} y \implies n(x) > n(y).$

Proof For part (i), (ii), and (iii) we assume, by Theorem 2.4.11 and Theorem 2.4.14, without loss of generality, that x is a basic term, and apply induction on the structure of basic terms. For part (iv) and (v) we again have to use induction on the general structure of terms.

(i). Case 1: x is an atomic action. Because $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models x \xrightarrow{a} \surd$, it must then be the case that $x \equiv \underline{\underline{a}}$. So we have $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta \vdash x = \underline{\underline{a}} = \underline{\underline{a}} + \underline{\underline{a}} = \underline{\underline{a}} + x$. Case 2: x is of the form atomic action followed by another basic term. This is in contradiction with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models x \xrightarrow{a} \surd$, so this case does not occur. Case 3: x is of the form $s + t$, where s and t are again basic terms. As $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models s + t \xrightarrow{a} \surd$, necessarily $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models s \xrightarrow{a} \surd$ or $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models t \xrightarrow{a} \surd$. Therefore, by the induction hypothesis, $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta \vdash s = \underline{\underline{a}} + s$ or $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta \vdash t = \underline{\underline{a}} + t$. But then in both cases $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta \vdash x = s + t = \underline{\underline{a}} + s + t = \underline{\underline{a}} + x$. Case 4: x is of the form $\sigma_{\mathrm{rel}}(s)$ for some basic term s. As we know that $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models x \xrightarrow{a} \surd$, this case cannot occur.

(ii). Case 1: x is an atomic action. This is in contradiction with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models x \xrightarrow{a} y$, so this case does not occur. Case 2: x is of the form atomic action followed by another basic term. Then, because $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models x \xrightarrow{a} y$, it must be that $x \equiv a \cdot y$. So, $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta \vdash x = a \cdot y = a \cdot y + a \cdot y = a \cdot y + x$. Case 3: x is of the form $s+t$, where s and t are again basic terms. As $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models s + t \xrightarrow{a} y$, necessarily $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models s \xrightarrow{a} y$ or $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models t \xrightarrow{a} y$. Therefore, by the induction hypothesis, $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta \vdash s = a \cdot y + s$ or $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta \vdash t = a \cdot y + t$. So in both cases $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta \vdash x = s + t = a \cdot y + s + t = a \cdot y + x$. Case 4: x is of the form $\sigma_{\mathrm{rel}}(s)$ for some basic term s. As we know that $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models x \xrightarrow{a} y$, this case cannot occur.

(iii). Case 1: x is an atomic action. This is in contradiction with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 2: x is of the form atomic action followed by another basic term. For the same reason, this case cannot occur either. Case 3: x is of the form $s+t$ where s and t are again basic terms. As $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models x \xrightarrow{\sigma} y$, we know that either $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models s \xrightarrow{\sigma} y$, or $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models t \xrightarrow{\sigma} y$, or both. So, by the induction hypothesis, either $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta \vdash t = \sigma_{\mathrm{rel}}(y) + t$, or $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta \vdash s = \sigma_{\mathrm{rel}}(y) + s$, or both. So in all cases $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta \vdash x = s + t = \sigma_{\mathrm{rel}}(y) + s + t = \sigma_{\mathrm{rel}}(y) + x$. Case 4: x is of the form $\sigma_{\mathrm{rel}}(s)$ for some basic term s. Then necessarily $s \equiv y$. So, $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta \vdash x = x + x = \sigma_{\mathrm{rel}}(y) + x = \sigma_{\mathrm{rel}}(s) + x$.

(iv). Case 1: x is an atomic action. This is in contradiction with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models x \xrightarrow{a} y$, so this case does not occur. Case 2: x is of the form $s \cdot t$, for certain terms s and t. Then, by $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models x \xrightarrow{a} y$, we either have $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models s \xrightarrow{a} \surd$ and $y \equiv t$, or we have $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models s \xrightarrow{a} s'$ and $y \equiv s' \cdot t$ for some term s'. In the first case, we have $n(x) = n(s \cdot t) = n(s) + n(t) + 1 > n(t) = n(y)$, and in the second we can apply the induction hypothesis to arrive at $n(s) > n(s')$, so we get $n(x) = n(s \cdot t) = n(s) + n(t) + 1 > n(s') + n(t) + 1 = n(s' \cdot t) = n(y)$. Case 3: x is of the form $s+t$, for certain terms s and t. As $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models s + t \xrightarrow{a} y$, necessarily $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models s \xrightarrow{a} y$ or $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models t \xrightarrow{a} y$. Therefore, by the induction hypothesis, $n(s) > n(y)$ or $n(t) > n(y)$. As n ranges over the positive naturals only, in both cases $n(x) = n(s + t) = n(s) + n(t) + 1 > n(y)$. Case 4: $x \equiv \sigma_{\mathrm{rel}}(s)$ for a certain term s, does not occur, as $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models \sigma_{\mathrm{rel}}(s) \stackrel{a}{\nrightarrow}$.

(v). Case 1: x is an atomic action. This is in contradiction with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 2: x is of the form $s \cdot t$, for certain terms s and t. Then, necessarily, $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models s \xrightarrow{\sigma} s'$ and $y \equiv s' \cdot t$ for some term s'. We now can apply the induction hypothesis to arrive at $n(s) > n(s')$, so we get $n(x) = n(s \cdot t) = n(s) + n(t) + 1 > n(s') + n(t) + 1 = n(s' \cdot t) = n(y)$. Case 3: x is of the form $s + t$, for certain terms s and t. Now, by $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models x \xrightarrow{\sigma} y$ we know that either $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models s \xrightarrow{\sigma} y$, or $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta) \models t \xrightarrow{\sigma} y$, or both. So, by the induction hypothesis, either $n(s) > n(y)$, or $n(t) > n(y)$, or both. So in all cases $n(x) = n(s + t) = n(s) + n(t) + 1 > n(y)$. Case 4: if x is of the form $\sigma_{\mathrm{rel}}(s)$, for a certain term s, it must be the case that $s \equiv y$. So, $n(x) = n(\sigma_{\mathrm{rel}}(s)) = n(\sigma_{\mathrm{rel}}(y)) = n(y) + 1 > n(y)$.

■

Theorem 2.4.17 (Completeness of $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta$)
The axiom system $\mathrm{BPA}^{-}_{drt}\text{-}\delta$ is a complete axiomatization of the set of closed $\mathrm{BPA}^{-}_{drt}\text{-}\delta$ terms modulo bisimulation equivalence.

Proof This proof is almost identical to the one of Theorem 2.2.22, with the exception of the fourth case. Let x and y be bisimilar closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ terms. We have to prove that $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta \vdash x = y$. With the aid of Theorem 2.2.16 and Theorem 2.4.14, it is enough to prove this for basic terms. By Corollary 2.2.21 it is even enough to prove for all basic terms x and y that:

$$x + y \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta} y \implies \mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta \vdash x + y = y.$$

We will prove this by induction on $n(x)$, using Lemma 2.4.16(iv)–(v) and case distinction on the form of basic term x. Case 1: x is of the form $\underline{\underline{a}}$, for $a \in A$. Then $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models x \xrightarrow{a} \surd$, so $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models x + y \xrightarrow{a} \surd$, and because $x + y \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta} y$ we have $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models y \xrightarrow{a} \surd$, so with Lemma 2.4.16(i) we find that $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta \vdash x + y = y$. This proves the basis of our induction. Case 2: x is of the form $\underline{\underline{a}} \cdot s$, when s is again a basic $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ term. Then $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models x \xrightarrow{a} s$, and therefore $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models x + y \xrightarrow{a} s$, so because $x + y \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta} y$ there is an s' with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models y \xrightarrow{a} s'$ and $s \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta} s'$. But then by Theorem 2.4.14 and Axiom A3 also $s + s' \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta} s'$ and $s' + s \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta} s$ and with induction we find $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta \vdash s + s' = s'$ and $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta \vdash s' + s = s$. So $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta \vdash s = s'$. Now $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta \vdash x + y = \underline{\underline{a}} \cdot s + y = \underline{\underline{a}} \cdot s' + y = y$ with Lemma 2.4.16(ii). Case 3: x is of the form $s + t$, for certain basic $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ terms s and t. Since $x + y \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta} y$, we also have $s + y \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta} y$ and $t + y \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta} y$. Then by the induction hypothesis $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta \vdash s + y = y$ and $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta \vdash t + y = y$. So $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta \vdash x + y = s + t + y = s + y = y$. Case 4: x is of the form $\sigma_{\mathrm{rel}}(x')$, for a certain basic $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ term x'. Now $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models x \xrightarrow{\sigma} x'$, and since $x + y \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta} y$, we also have $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models y \xrightarrow{\sigma} y'$ and $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta) \models x + y \xrightarrow{\sigma} x' + y'$ for some y' such that $x' + y' \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta} y'$. By Lemma 2.4.16(iii) we have $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta \vdash y = \sigma_{\mathrm{rel}}(y') + y$. By the induction hypothesis we have $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta \vdash x' + y' = y'$. So, $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta \vdash x + y = \sigma_{\mathrm{rel}}(x') + y = \sigma_{\mathrm{rel}}(x') + \sigma_{\mathrm{rel}}(y') + y = \sigma_{\mathrm{rel}}(x' + y') + y = \sigma_{\mathrm{rel}}(y') + y = y$. ■

Remark 2.4.18 (Completeness of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$)
Completeness of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ is also claimed in Theorem 2.12.5 of [13] (where $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ is called $\mathrm{BPA}_{\mathrm{dt}}$). The proof given there, however, is incorrect: the (supposedly) bijective mapping φ is not bijective, as $\varphi^{-1}(\sigma)$ is undefined.

2.5 Soundness and Completeness of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID

Definition 2.5.1 (Signature of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID)
The signature of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID consists of the *undelayable atomic actions* $\{\underline{\underline{a}} | a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *time unit delay operator* σ_{rel}, and the *"now" operator* ν_{rel}.

Definition 2.5.2 (Axioms of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID)
The process algebra $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID is axiomatized by the axioms of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ given in Definition 2.4.2 on page 13, Axioms DRT3–DRT5 shown in Table 9 on the following page, and Axioms DCS1–DCS4 shown in Table 11 on the next page: $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID = A1–A5 + DRT1–DRT5 + DCS1–DCS4.

Remark 2.5.3 (DRT4 and DRT5 versus DRT4A)
Note that for *closed* $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID terms Axioms DRT4–DRT5 given in Table 9 on the following page are equivalent with Axiom DRT4A given in Table 10 on the next page. Therefore we could replace Axioms DRT4–DRT5 in $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID by DRT4A without affecting the soundness or completeness of the resulting theory. This is for example done in [16].

$$\underline{\underline{\delta}} \cdot x = \underline{\underline{\delta}} \qquad \text{DRT3}$$
$$\underline{\underline{a}} + \underline{\underline{\delta}} = \underline{\underline{a}} \qquad \text{DRT4}$$
$$\sigma_{\mathrm{rel}}(x) + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(x) \qquad \text{DRT5}$$

Table 9: Axioms for $\underline{\underline{\delta}}$.

One such reason to do so, could be the fact that DRT4A is a straightforward reformulation of Axiom A6 from BPA_δ, in a setting with undelayable actions. However, we prefer Axioms DRT4–DRT5 over DRT4A because the latter does not extend to the discrete-time theories with immediate deadlock we will describe in the sections to follow. If we need the equality of DRT4A, we can derive it as a lemma (which we do in Lemma 2.5.16(iv)). We will return to this subject in Remark 2.7.3 on page 45.

$$x + \underline{\underline{\delta}} = x \qquad \text{DRT4A}$$

Table 10: Alternative for Axioms DRT4–DRT5.

$$\nu_{\mathrm{rel}}(\underline{\underline{a}}) = \underline{\underline{a}} \qquad \text{DCS1}$$
$$\nu_{\mathrm{rel}}(x + y) = \nu_{\mathrm{rel}}(x) + \nu_{\mathrm{rel}}(y) \qquad \text{DCS2}$$
$$\nu_{\mathrm{rel}}(x \cdot y) = \nu_{\mathrm{rel}}(x) \cdot y \qquad \text{DCS3}$$
$$\nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x)) = \underline{\underline{\delta}} \qquad \text{DCS4}$$

Table 11: Axioms for ν_{rel}.

Definition 2.5.4 (Semantics of $\mathrm{BPA}^-_{\mathrm{drt}}$–ID)
The semantics of $\mathrm{BPA}^-_{\mathrm{drt}}$–ID are given by the term deduction system $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID})$, induced by the deduction rules for $\mathrm{BPA}^-_{\mathrm{drt}}$–$\delta$ given in Definition 2.4.4 on page 14, and the deduction rules for ν_{rel} shown in Table 12 on the following page.

Definition 2.5.5 (Bisimulation and Bisimulation Model for $\mathrm{BPA}^-_{\mathrm{drt}}$–ID)
Bisimulation for $\mathrm{BPA}^-_{\mathrm{drt}}$–ID and the corresponding bisimulation model are defined in the same way as for $\mathrm{BPA}^-_{\mathrm{drt}}$–$\delta$ and BPA respectively. Replace “$\mathrm{BPA}^-_{\mathrm{drt}}$–$\delta$” by “$\mathrm{BPA}^-_{\mathrm{drt}}$–ID” in Definition 2.4.5 on page 14 and “BPA” by “$\mathrm{BPA}^-_{\mathrm{drt}}$–ID” in Definition 2.2.11 on page 8.

Definition 2.5.6 (Basic Terms of $\mathrm{BPA}^-_{\mathrm{drt}}$–ID)
We define $(\sigma, \underline{\underline{\delta}})$*-basic terms* inductively as follows:

$$\frac{x \xrightarrow{a} x'}{\nu_{\mathrm{rel}}(x) \xrightarrow{a} x'} \qquad \frac{x \xrightarrow{a} \surd}{\nu_{\mathrm{rel}}(x) \xrightarrow{a} \surd}$$

Table 12: Deduction rules for ν_{rel}.

(i). For every $a \in A_\delta$, $\underline{\underline{a}}$ is a $(\sigma, \underline{\underline{\delta}})$-basic term,

(ii). if $a \in A_\delta$ and t is a $(\sigma, \underline{\underline{\delta}})$-basic term, then $\underline{\underline{a}} \cdot t$ is a $(\sigma, \underline{\underline{\delta}})$-basic term,

(iii). if t and s are $(\sigma, \underline{\underline{\delta}})$-basic terms, then $t + s$ is a $(\sigma, \underline{\underline{\delta}})$-basic term,

(iv). if t is a $(\sigma, \underline{\underline{\delta}})$-basic term, then $\sigma_{\mathrm{rel}}(t)$ is a $(\sigma, \underline{\underline{\delta}})$-basic term.

From now on, if we speak of basic terms in the context of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID, we mean $(\sigma, \underline{\underline{\delta}})$-basic terms.

Definition 2.5.7 (Number of Symbols of a $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID Term)
We define $n(x)$, the number of symbols of x, inductively as follows:

(i). For $a \in A_\delta$, we define $n(\underline{\underline{a}}) = 1$,

(ii). for closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID terms x and y, we define $n(x+y) = n(x \cdot y) = n(x) + n(y) + 1$,

(iii). for a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term x, we define $n(\sigma_{\mathrm{rel}}(x)) = n(\nu_{\mathrm{rel}}(x)) = n(x) + 1$.

Definition 2.5.8 (Summation Convention)
We will use the convention that a summation over the empty set yields the undelayable deadlock:

$$\sum_{i \in \emptyset} t_i = \underline{\underline{\delta}}.$$

Theorem 2.5.9 (General Form of Basic Terms of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID)
Modulo the commutativity and associativity of the $+$, all basic terms t of BPA^{-}_{drt}-ID are of the form:

$$t \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \sigma_{rel}(u_k)$$

for $m, n, p \in \mathbb{N}$, $a_i, b_j \in A_\delta$, and basic terms s_i and u_k.

Proof Trivial, by inspection of the definition of basic terms, Definition 2.5.6. Observe that the general form of basic terms is closed under the formation rules gives in Definition 2.5.6. See also [10]. ■

Lemma 2.5.10 (Representation of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID Terms)
Let t be a basic term. Then either BPA^{-}_{drt}-ID $\vdash t = \nu_{rel}(t)$, or there exists a basic term s such that BPA^{-}_{drt}-ID $\vdash t = \nu_{rel}(t) + \sigma_{rel}(s)$ and $n(s) < n(t)$.

Proof Let t be a basic term. By Theorem 2.5.9, we may now proceed by case analysis on the general form of basic terms:

(i). Either we have no σ_{rel}-summands ($p = 0$ in Definition 2.5.9):

$$t \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}}$$

for $m, n \in \mathbb{N}$, $a_i, b_j \in A_\delta$, and basic terms s_i. Then we have the following computation:

$$\begin{aligned}
\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash t &= \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} \\
&= \sum_{i<m} \nu_{\mathrm{rel}}(\underline{\underline{a_i}}) \cdot s_i + \sum_{j<n} \nu_{\mathrm{rel}}(\underline{\underline{b_j}}) \\
&= \sum_{i<m} \nu_{\mathrm{rel}}(\underline{\underline{a_i}} \cdot s_i) + \sum_{j<n} \nu_{\mathrm{rel}}(\underline{\underline{b_j}}) \\
&= \nu_{\mathrm{rel}}\left(\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} \right) \\
&= \nu_{\mathrm{rel}}(t)
\end{aligned}$$

(ii). Or we have at least one σ_{rel}-summand ($p \geq 1$ in Definition 2.5.9) :

$$t \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \sigma_{\mathrm{rel}}(u_k)$$

for $m, n, p \in \mathbb{N}$, $a_i, b_j \in A_\delta$, and basic terms s_i and u_k. Then we have the following computation:

$$\begin{aligned}
\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash t &= \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \sigma_{\mathrm{rel}}(u_k) \\
&= \sum_{i<m} \nu_{\mathrm{rel}}(\underline{\underline{a_i}}) \cdot s_i + \sum_{j<n} \nu_{\mathrm{rel}}(\underline{\underline{b_j}}) + \sum_{k<p} \sigma_{\mathrm{rel}}(u_k) \\
&= \sum_{i<m} \nu_{\mathrm{rel}}(\underline{\underline{a_i}} \cdot s_i) + \sum_{j<n} \nu_{\mathrm{rel}}(\underline{\underline{b_j}}) + \sum_{k<p} \sigma_{\mathrm{rel}}(u_k) \\
&= \nu_{\mathrm{rel}}\left(\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} \right) + \sigma_{\mathrm{rel}}\left(\sum_{k<p} u_k \right) \\
&= \nu_{\mathrm{rel}}\left(\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \sigma_{\mathrm{rel}}(u_k) \right) + \sigma_{\mathrm{rel}}\left(\sum_{k<p} u_k \right) \\
&= \nu_{\mathrm{rel}}(t) + \sigma_{\mathrm{rel}}(s)
\end{aligned}$$

Where we define:

$$s \equiv \sum_{k<p} u_k$$

Note that $n(s) < n(t)$ is now trivially satisfied, as for every summand u_k of s, there is a corresponding summand $\sigma_{\mathrm{rel}}(u_k)$ of t, and at least one such summand exists as $p \geq 1$.

■

Remark 2.5.11 (Representation of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID Terms)
The main use of Lemma 2.5.10 will be in induction proofs regarding the (not yet treated) theories $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID and $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID (see Sections 3.2 to 3.5).

Theorem 2.5.12 (Elimination for $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID)
Let t be a closed BPA^{-}_{drt}–ID term. Then there is a basic term s such that $\mathrm{BPA}^{-}_{drt}\text{–ID} \vdash s = t$.

Proof This theorem is proven as follows. First a number of axioms of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID are selected, and subsequently oriented as rewriting rules. This gives us a term rewriting system. Then it is proven that this term rewriting system is strongly normalizing and that every normal form of a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term is a basic term. In this way a recipe is obtained for transforming a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term into a basic term.

The rewriting rules of the term rewriting system for $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID are given in Table 13. The proof that this term rewriting system is strongly normalizing uses the method of

$(x + y) \cdot z \to x \cdot z + y \cdot z$	RA4
$(x \cdot y) \cdot z \to x \cdot (y \cdot z)$	RA5
$\sigma_{\mathrm{rel}}(x) \cdot y \to \sigma_{\mathrm{rel}}(x \cdot y)$	RDRT2
$\nu_{\mathrm{rel}}(\underline{\underline{a}}) \to \underline{\underline{a}}$	RDCS1
$\nu_{\mathrm{rel}}(x + y) \to \nu_{\mathrm{rel}}(x) + \nu_{\mathrm{rel}}(y)$	RDCS2
$\nu_{\mathrm{rel}}(x \cdot y) \to \nu_{\mathrm{rel}}(x) \cdot y$	RDCS3
$\nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x)) \to \underline{\underline{\delta}}$	RDCS4

Table 13: Term Rewriting System for $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID.

the lexicographical path ordering.

The well-founded ordering $>$ on constants and function symbols is the following:

$$\nu_{\mathrm{rel}} > \cdot > + > \sigma_{\mathrm{rel}} > \underline{\underline{a}}$$

Moreover, $\cdot$ has the lexicographical status of the first argument. Now we show that the left-hand side of every rewriting rule is bigger than the right-hand side with respect to the ordering $\succ_{\mathrm{lpo}}$. This is done by the following reductions:

$$\begin{aligned}
\nu_{\mathrm{rel}}(\underline{\underline{a}}) &\succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}{}^{\star}(\underline{\underline{a}}) \\
&\succ_{\mathrm{lpo}} \underline{\underline{a}} \\
\nu_{\mathrm{rel}}(x + y) &\succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}{}^{\star}(x + y) \succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}{}^{\star}(x + y) + \nu_{\mathrm{rel}}{}^{\star}(x + y) \\
&\succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}(x +^{\star} y) + \nu_{\mathrm{rel}}(x +^{\star} y) \succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}(x) + \nu_{\mathrm{rel}}(y) \\
\nu_{\mathrm{rel}}(x \cdot y) &\succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}{}^{\star}(x \cdot y) \succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}{}^{\star}(x \cdot y) \cdot \nu_{\mathrm{rel}}{}^{\star}(x \cdot y) \succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}(x \cdot^{\star} y) \cdot (x \cdot y) \\
&\succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}(x) \cdot (x \cdot y) \succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}(x) \cdot^{\star} (x \cdot y) \succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}(x) \cdot (x \cdot^{\star} y) \succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}(x) \cdot y \\
\nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x)) &\succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}{}^{\star}(\sigma_{\mathrm{rel}}(x)) \\
&\succ_{\mathrm{lpo}} \underline{\underline{\delta}}
\end{aligned}$$

Note that we do not give reductions for RA4, RA5, and RDRT2 as these already have been given in the proof of Theorem 2.4.11, and since the new ordering is a proper extension of the old one, these proofs remain valid.

Next, we will prove that the normal forms of the closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID terms are basic terms. Thereto, suppose that s is a normal form of some closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term. Furthermore, suppose that s is not a basic term. Let s' denote the smallest subterm of s which is not a basic term. Then we can prove that s' is not a normal form by case analysis. We distinguish all possible cases:

(i). s' is an atomic action or $\underline{\underline{\delta}}$. But then s' is a basic term. This is in contradiction with the assumption that s' is not a basic term, so this case does not occur.

(ii). s' is of the form $s'_1 \cdot s'_2$ for basic terms s'_1 and s'_2. With case analysis on the structure of basic term s'_1:

- (a) If s'_1 is an atomic action or $\underline{\underline{\delta}}$, then $s'_1 \cdot s'_2$ is a basic term, and so s' is a basic term which again contradicts the assumption that s' is not a basic term. This case can therefore not occur.
- (b) If s'_1 is of the form $\underline{\underline{a}} \cdot t$ for some $a \in A_\delta$ and basic term t, then rewriting rule RA5 can be applied. So, s' is not a normal form.
- (c) If s'_1 is of the form $t_1 + t_2$ for t_1 and t_2 basic term s. Then rewriting rule RA4 is applicable. Therefore, s' is not a normal form.
- (d) If s'_1 is of the form $\sigma_{\mathrm{rel}}(t)$ for some basic term t. Then rewriting rule RDRT2 is applicable. So, s' is not a normal form.

(iii). s' is of the form $s'_1 + s'_2$ for basic terms s'_1 and s'_2. In this case s' would be a basic term, which contradicts the assumption that s' is not a basic term. Therefore, this case cannot occur.

(iv). s' is of the form $\sigma_{\mathrm{rel}}(t)$ for some basic term t. But then s' is basic term too, so the case does not occur.

(v). s' is of the form $\nu_{\mathrm{rel}}(t)$ for some basic term t. But then one of RDCS1–RDCS4 is applicable, so s' is not a normal form.

In any case that can occur it follows that s' is not a normal form. Since s' is a subterm of s, we conclude that s is not a normal form. This contradicts the assumption that s is a normal form. From this contradiction we conclude that s is a basic term, which completes the proof. ■

Remark 2.5.13 (Elimination for $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID)
Elimination for $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID is also claimed (without proof) in Theorem 2.1 of [11].

Theorem 2.5.14 (Soundness of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID)
The set of closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$-ID terms modulo bisimulation equivalence is a model of $\mathrm{BPA}^{-}_{\mathrm{drt}}$-ID.

Proof In Theorem 2.4.14 we already proved the soundness of Axioms A1–A5 and DRT1–DRT2 with respect to the term deduction system $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta)$. Since the term deduction system $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID})$ uses the same underlying model as the term deduction system $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–}\delta)$, these proofs remain valid in the setting of $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID. Therefore, we only have to prove soundness of the additional Axioms DRT3–DRT5 and DCS1–DCS4.

Axiom DRT3 Take the relation:

$$R = \{(s, s), (s + \underline{\underline{\delta}}, s) \mid s \in C(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID})\}$$

We look at the transitions of both sides at the same time. First, $s + \underline{\underline{\delta}} \xrightarrow{a} p$ iff $s \xrightarrow{a} p$, and note that $(p, p) \in R$. Secondly, $s + \underline{\underline{\delta}} \xrightarrow{a} \surd$ iff $s \xrightarrow{a} \surd$. Thirdly, $s + \underline{\underline{\delta}} \xrightarrow{\sigma} p$ iff $s \xrightarrow{\sigma} p$, and note that $(p, p) \in R$.

Axiom DRT4 Take the relation:

$$R = \{(\underline{\underline{\delta}} \cdot s, \underline{\underline{\delta}}) \mid s \in C(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID})\}$$

We look at the transitions of both sides at the same time. Observe that neither the left-hand side nor the right-hand side of the axiom can perform any transition: $\underline{\underline{\delta}} \cdot s \stackrel{a}{\nrightarrow}$, $\underline{\underline{\delta}} \cdot s \stackrel{\sigma}{\nrightarrow}$ and $\underline{\underline{\delta}} \stackrel{a}{\nrightarrow}$, $\underline{\underline{\delta}} \stackrel{\sigma}{\nrightarrow}$.

Axiom DRT5 Take the relation:

$$R = \{(s, s), (\sigma_{\mathrm{rel}}(s) + \underline{\underline{\delta}}, \sigma_{\mathrm{rel}}(s)) \mid s \in C(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID})\}$$

We look at the transitions of both sides at the same time. Observe that neither the left-hand side nor the right-hand side of the axiom can perform an a-transition: $\sigma_{\mathrm{rel}}(s) + \underline{\underline{\delta}} \stackrel{a}{\nrightarrow}$ and $\sigma_{\mathrm{rel}}(s) \stackrel{a}{\nrightarrow}$. Furthermore, the only σ-transitions are $\sigma_{\mathrm{rel}}(s) + \underline{\underline{\delta}} \xrightarrow{\sigma} s$ and $\sigma_{\mathrm{rel}}(s) \xrightarrow{\sigma} s$, and note that $(s, s) \in R$.

Axiom DCS1 Take the relation:

$$R = \{(\nu_{\mathrm{rel}}(\underline{\underline{a}}), \underline{\underline{a}})\}$$

We look at the transitions of both sides at the same time. Observe that either side of the axiom can only do an a-transition to $\surd$: $\nu_{\mathrm{rel}}(\underline{\underline{a}}) \xrightarrow{a} \surd$ and $\underline{\underline{a}} \xrightarrow{a} \surd$. No other transitions are possible.

Axiom DCS2 Take the relation:

$$R = \{(s, s), (\nu_{\mathrm{rel}}(s + t), \nu_{\mathrm{rel}}(s) + \nu_{\mathrm{rel}}(t)) \mid s, t \in C(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID})\}$$

We look at the transitions of both sides at the same time. Observe that neither side of the axiom can do a σ-transition: $\nu_{\mathrm{rel}}(s + t) \stackrel{\sigma}{\nrightarrow}$ and $\nu_{\mathrm{rel}}(s) + \nu_{\mathrm{rel}}(t) \stackrel{\sigma}{\nrightarrow}$. Furthermore, $\nu_{\mathrm{rel}}(s + t) \xrightarrow{a} p$ iff $s + t \xrightarrow{a} p$ iff $s \xrightarrow{a} p$ or $t \xrightarrow{a} p$ iff $\nu_{\mathrm{rel}}(s) \xrightarrow{a} p$ or $\nu_{\mathrm{rel}}(t) \xrightarrow{a} p$ iff $\nu_{\mathrm{rel}}(s) + \nu_{\mathrm{rel}}(t) \xrightarrow{a} p$, and note that $(p, p) \in R$. Finally, $\nu_{\mathrm{rel}}(s + t) \xrightarrow{a} \surd$ iff $s + t \xrightarrow{a} \surd$ iff $s \xrightarrow{a} \surd$ or $t \xrightarrow{a} \surd$ iff $\nu_{\mathrm{rel}}(s) \xrightarrow{a} \surd$ or $\nu_{\mathrm{rel}}(t) \xrightarrow{a} \surd$ iff $\nu_{\mathrm{rel}}(s) + \nu_{\mathrm{rel}}(t) \xrightarrow{a} \surd$.

Axiom DCS3 Take the relation:

$$R = \{(s, s), (\nu_{\mathrm{rel}}(s \cdot t), \nu_{\mathrm{rel}}(s) \cdot t) \mid s, t \in C(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID})\}$$

We look at the transitions of both sides at the same time. Observe that neither side of the axiom can do a σ-transition: $\nu_{\mathrm{rel}}(s \cdot t) \stackrel{\sigma}{\nrightarrow}$ and $\nu_{\mathrm{rel}}(s) \cdot t \stackrel{\sigma}{\nrightarrow}$. Furthermore, $\nu_{\mathrm{rel}}(s \cdot t) \xrightarrow{a} p$ iff $s \cdot t \xrightarrow{a} p$ iff $s \xrightarrow{a} \surd$ and $p \equiv t$ or $s \xrightarrow{a} s'$ and $p \equiv s' \cdot t$ iff $\nu_{\mathrm{rel}}(s) \xrightarrow{a} \surd$ and $p \equiv t$ or $\nu_{\mathrm{rel}}(s) \xrightarrow{a} s'$ and $p \equiv s' \cdot t$ iff $\nu_{\mathrm{rel}}(s) \cdot t \xrightarrow{a} p$, and note that $(p, p) \in R$. Finally, $\nu_{\mathrm{rel}}(s \cdot t) \stackrel{a}{\nrightarrow} \surd$ and $\nu_{\mathrm{rel}}(s) \cdot t \stackrel{a}{\nrightarrow} \surd$.

Axiom DCS4 Take the relation:

$$R = \{(\nu_{\text{rel}}(\sigma_{\text{rel}}(s)), \underline{\underline{\delta}}) \mid s \in C(\text{BPA}^-_{\text{drt}}\text{-ID})\}$$

We look at the transitions of both sides at the same time. Observe that neither the left-hand side nor the right-hand side of the axiom can perform any a-transition or σ-transition: $\nu_{\text{rel}}(\sigma_{\text{rel}}(s)) \stackrel{a}{\nrightarrow}$, $\nu_{\text{rel}}(\sigma_{\text{rel}}(s)) \stackrel{\sigma}{\nrightarrow}$, and $\underline{\underline{\delta}} \stackrel{a}{\nrightarrow}$, $\underline{\underline{\delta}} \stackrel{\sigma}{\nrightarrow}$.

■

Remark 2.5.15 (Soundness of BPA$^-_{\text{drt}}$-ID)
Soundness of BPA$^-_{\text{drt}}$-ID is also claimed (without proof) in Section 2.12.1 of [13] (where BPA$^-_{\text{drt}}$-ID is called BPA$_{\delta\text{dt}}$), and in Theorem 2.2 of [11].

Lemma 2.5.16 (Towards Completeness of BPA$^-_{\text{drt}}$-ID)
Let x be a closed BPA$^-_{drt}$-ID term and let $a \in A$. Then we have:

(i). $T(BPA^-_{drt}\text{-}ID) \models x \xrightarrow{a} \surd \implies BPA^-_{drt}\text{-}ID \vdash x = \underline{\underline{a}} + x$,

(ii). $T(BPA^-_{drt}\text{-}ID) \models x \xrightarrow{a} y \implies BPA^-_{drt}\text{-}ID \vdash x = \underline{\underline{a}} \cdot y + x$,

(iii). $T(BPA^-_{drt}\text{-}ID) \models x \stackrel{\sigma}{\nrightarrow} \implies BPA^-_{drt}\text{-}ID \vdash x = \nu_{rel}(x)$,

(iv). $BPA^-_{drt}\text{-}ID \vdash x + \underline{\underline{\delta}} = x$,

(v). $T(BPA^-_{drt}\text{-}ID) \models x \xrightarrow{\sigma} y \implies BPA^-_{drt}\text{-}ID \vdash x = \sigma_{rel}(y) + \nu_{rel}(x)$,

(vi). $T(BPA^-_{drt}\text{-}ID) \models x \xrightarrow{a} y \implies n(x) > n(y)$,

(vii). $T(BPA^-_{drt}\text{-}ID) \models x \xrightarrow{\sigma} y \implies n(x) > n(y)$.

Proof For part (i)–(v) we assume, by Theorem 2.5.12 and Theorem 2.5.14, without loss of generality, that x is a basic term, and then apply induction on the structure of basic terms. For part (vi) and (vii) we again have to use induction on the general structure of terms.

(i). Suppose that $T(\text{BPA}^-_{\text{drt}}\text{-ID}) \models x \xrightarrow{a} \surd$. Case 1: $x \equiv \underline{\underline{b}}$, where $b \in A_\delta$. Because $T(\text{BPA}^-_{\text{drt}}\text{-ID}) \models x \xrightarrow{a} \surd$, it must be the case that $b \equiv a$. So we have $\text{BPA}^-_{\text{drt}}\text{-ID} \vdash x = \underline{\underline{b}} = \underline{\underline{b}} + \underline{\underline{b}} = \underline{\underline{a}} + \underline{\underline{b}} = \underline{\underline{a}} + x$. Case 2: $x \equiv \underline{\underline{b}} \cdot x'$, where $b \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\text{BPA}^-_{\text{drt}}\text{-ID}) \models x \xrightarrow{a} \surd$, so this case does not occur. Case 3: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\text{BPA}^-_{\text{drt}}\text{-ID}) \models x \xrightarrow{a} \surd$, necessarily $T(\text{BPA}^-_{\text{drt}}\text{-ID}) \models x' \xrightarrow{a} \surd$ or $T(\text{BPA}^-_{\text{drt}}\text{-ID}) \models x'' \xrightarrow{a} \surd$. Therefore, by the induction hypothesis, $\text{BPA}^-_{\text{drt}}\text{-ID} \vdash x' = \underline{\underline{a}} + x'$ or $\text{BPA}^-_{\text{drt}}\text{-ID} \vdash x'' = \underline{\underline{a}} + x''$. But then in both cases $\text{BPA}^-_{\text{drt}}\text{-ID} \vdash x = x' + x'' = \underline{\underline{a}} + x' + x'' = \underline{\underline{a}} + x$. Case 4: $x \equiv \sigma_{\text{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\text{BPA}^-_{\text{drt}}\text{-ID}) \models x \xrightarrow{a} \surd$, so this case does not occur.

(ii). Suppose that $T(\text{BPA}^-_{\text{drt}}\text{-ID}) \models x \xrightarrow{a} y$. Case 1: $x \equiv \underline{\underline{b}}$, where $b \in A_\delta$. This is in contradiction with $T(\text{BPA}^-_{\text{drt}}\text{-ID}) \models x \xrightarrow{a} y$, so this case does not occur. Case 2: $x \equiv \underline{\underline{b}} \cdot x'$, where $b \in A_\delta$ and x' is a basic term. Then, because $T(\text{BPA}^-_{\text{drt}}\text{-ID}) \models x \xrightarrow{a} y$, it must be that $b \equiv a$ and $x' \equiv y$. So, $\text{BPA}^-_{\text{drt}}\text{-ID} \vdash x = x + x = \underline{\underline{b}} \cdot x' + x = \underline{\underline{a}} \cdot y + x$. Case 3: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\text{BPA}^-_{\text{drt}}\text{-ID}) \models x \xrightarrow{a} y$, necessarily

$T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x' \xrightarrow{a} y$ or $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x'' \xrightarrow{a} y$. Therefore, by the induction hypothesis, $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x' = \underline{\underline{a}} \cdot y + x'$ or $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x'' = \underline{\underline{a}} \cdot y + x''$. But then in both cases $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x = x' + x'' = \underline{\underline{a}} \cdot y + x' + x'' = \underline{\underline{a}} \cdot y + x$. Case 4: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{a} y$, so this case does not occur.

(iii). Suppose that $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x \overset{\sigma}{\not\to}$. Case 1: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. We have $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x = \underline{\underline{a}} = \nu_{\mathrm{rel}}(\underline{\underline{a}}) = \nu_{\mathrm{rel}}(x)$. Case 2: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. We have $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x = \underline{\underline{a}} \cdot x' = \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot x' = \nu_{\mathrm{rel}}(\underline{\underline{a}} \cdot x') = \nu_{\mathrm{rel}}(x)$. Case 3: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x \overset{\sigma}{\not\to}$, necessarily $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x' \overset{\sigma}{\not\to}$ and $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x'' \overset{\sigma}{\not\to}$. Therefore, by the induction hypothesis, $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x' = \nu_{\mathrm{rel}}(x')$ and $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x'' = \nu_{\mathrm{rel}}(x'')$. But then also $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x = x' + x'' = \nu_{\mathrm{rel}}(x') + \nu_{\mathrm{rel}}(x'') = \nu_{\mathrm{rel}}(x' + x'') = \nu_{\mathrm{rel}}(x)$. Case 4: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x \overset{\sigma}{\not\to}$, so this case does not occur.

(iv). Case 1: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. Then we have $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x + \underline{\underline{\delta}} = \underline{\underline{a}} + \underline{\underline{\delta}} = \underline{\underline{a}} = x$. Case 2: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. Then $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x + \underline{\underline{\delta}} = \underline{\underline{a}} \cdot x' + \underline{\underline{\delta}} = \underline{\underline{a}} \cdot x' + \underline{\underline{\delta}} \cdot x' = (\underline{\underline{a}} + \underline{\underline{\delta}}) \cdot x' = \underline{\underline{a}} \cdot x' = x$. Case 3: $x \equiv x' + x''$, where x' and x'' are basic terms. Then, by the induction hypothesis, $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x' + \underline{\underline{\delta}} = x', x'' + \underline{\underline{\delta}} = x''$. So, $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x + \underline{\underline{\delta}} = x' + x'' + \underline{\underline{\delta}} = x' + x'' = x$. Case 4: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. Then $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(x') + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(x') = x$.

(v). Suppose that $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$. Case 1: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 2: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 3: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, necessarily (1) $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x' \xrightarrow{\sigma} y, x'' \overset{\sigma}{\not\to}$, or, (2) $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x' \overset{\sigma}{\not\to}, x'' \xrightarrow{\sigma} y$, or, (3) $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x' \xrightarrow{\sigma} y', x'' \xrightarrow{\sigma} y''$ where $y \equiv y' + y''$. In the first case, by the induction hypothesis, we have $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x')$, and, by (iii), $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x'' = \nu_{\mathrm{rel}}(x'')$. Therefore, $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x = x' + x'' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x') + \nu_{\mathrm{rel}}(x'') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x' + x'') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$. The second case is treated analogously. In the third case we have, by the induction hypothesis, $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x' = \sigma_{\mathrm{rel}}(y') + \nu_{\mathrm{rel}}(x'), x'' = \sigma_{\mathrm{rel}}(y'') + \nu_{\mathrm{rel}}(x'')$. Therefore we have $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x = x' + x'' = \sigma_{\mathrm{rel}}(y') + \nu_{\mathrm{rel}}(x') + \sigma_{\mathrm{rel}}(y'') + \nu_{\mathrm{rel}}(x'') = \sigma_{\mathrm{rel}}(y' + y'') + \nu_{\mathrm{rel}}(x' + x'') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$. Case 4: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. Because $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, it must be the case that $x' \equiv y$. So we have $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash x = \sigma_{\mathrm{rel}}(x') = \sigma_{\mathrm{rel}}(y) = \sigma_{\mathrm{rel}}(y) + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x')) = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$.

(vi). Suppose that $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{a} y$. Case 1: $x \equiv \underline{\underline{b}}$, where $b \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{a} y$, so this case does not occur. Case 2: $x \equiv x' \cdot x''$, for certain terms x' and x''. Then, because $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{a} y$, we either have $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x' \xrightarrow{a} \surd$ and $y \equiv x''$, or we have $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID}) \models x' \xrightarrow{a} x'''$ and $y \equiv x''' \cdot x''$ for some term x'''. In the first case, we have $n(x) = n(x' \cdot x'') = n(x') + n(x'') + 1 > n(x'') = n(y)$, and in the second we can apply the induction hypothesis to arrive at $n(x') > n(x''')$, so we get $n(x) = n(x' \cdot x'') = n(x') + n(x'') + 1 > n(x''') + n(x') + 1 = n(x''' \cdot x'') = n(y)$. Case 3: $x \equiv x' + x''$, for certain terms x' and x''. Since $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{a} y$, necessarily $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x' \xrightarrow{a} y$ or $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x'' \xrightarrow{a} y$. Therefore, by the induction hypothesis, $n(x') > n(y)$ or $n(x'') > n(y)$. In both cases $n(x) = n(x' + x'') = n(x') + n(x'') + 1 > n(y)$. Case 4: $x \equiv \sigma_{\mathrm{rel}}(x')$, for a certain term x'. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{a} y$, so this case does not occur. Case 5: $x \equiv \nu_{\mathrm{rel}}(x')$, for a certain term x'. Since $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{a} y$, necessarily $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x' \xrightarrow{a} y$. Therefore, by the induction hypothesis, $n(x') > n(y)$. So, $n(x) = n(\nu_{\mathrm{rel}}(x')) = n(x') + 1 > n(y)$.

(vii). Suppose that $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$. Case 1: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 2: $x \equiv x' \cdot x''$, for certain terms x' and x''. Then necessarily, $x' \xrightarrow{\sigma} x'''$ and $y \equiv x''' \cdot x''$ for some term x'''. We now can apply the induction hypothesis to arrive at $n(x') > n(x''')$, so we get $n(x) = n(x' \cdot x'') = n(x') + n(x'') + 1 > n(x''') + n(x'') + 1 = n(x''' \cdot x'') = n(y)$. Case 3: $x \equiv x' + x''$, for certain terms x' and x''. As $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, necessarily (1) $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x' \xrightarrow{\sigma} y, x'' \overset{\sigma}{\nrightarrow}$, or, (2) $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x' \overset{\sigma}{\nrightarrow}, x'' \xrightarrow{\sigma} y$, or, (3) $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x' \xrightarrow{\sigma} y', x'' \xrightarrow{\sigma} y''$ where $y \equiv y' + y''$. In the first case, by the induction hypothesis, $n(x') > n(y)$. So $n(x) = n(x' + x'') = n(x') + n(x'') + 1 > n(y)$. The second case is treated analogously. In the third case, by the induction hypothesis, $n(x') > n(y')$ and $n(x'') > n(y'')$. So $n(x) = n(x' + x'') = n(x') + n(x'') + 1 > n(y') + n(y'') + 1 = n(y)$. Case 4: $x \equiv \sigma_{\mathrm{rel}}(x')$, for a certain term x'. Because $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, it must be the case that $x' \equiv y$. Then we have $n(x) = n(\sigma_{\mathrm{rel}}(x')) = n(x') + 1 = n(y) + 1 > n(y)$. Case 5: $x \equiv \nu_{\mathrm{rel}}(x')$, for a certain term x'. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, so this case does not occur.

■

Theorem 2.5.17 (Completeness of $\mathbf{BPA}^-_{\mathbf{drt}}$–ID)
The axiom system BPA^-_{drt}-ID is a complete axiomatization of the set of closed BPA^-_{drt}-ID terms modulo (strong) bisimulation equivalence.

Proof Suppose $x + y \sim_{\mathrm{BPA}^-_{\mathrm{drt}}\text{-ID}} y$. We will prove, with induction on the structure of basic term x, that $\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID} \vdash x + y = y$. By Theorem 2.5.12 we can restrict ourselves to basic terms without loss of generality. The proof is done with induction on $n(x)$, using Lemma 2.5.16(vi)–(vii) and case distinction on the form of basic term x.

(i). $x \equiv \underline{\underline{\delta}}$. Then, using Lemma 2.5.16(iv) we have $\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID} \vdash x + y = \underline{\underline{\delta}} + y = y + \underline{\underline{\delta}} = y$.

(ii). $x \equiv \underline{\underline{a}}$, where $a \in A$. From the deduction rules we have $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{a} \surd$ and $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x + y \xrightarrow{a} \surd$. Since $x + y \sim_{\mathrm{BPA}^-_{\mathrm{drt}}\text{-ID}} y$ we also have $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models y \xrightarrow{a} \surd$. By Lemma 2.5.16(i) we obtain $\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID} \vdash y = \underline{\underline{a}} + y$. So, $\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID} \vdash x + y = \underline{\underline{a}} + y = y$.

(iii). $x \equiv \underline{\underline{\delta}} \cdot s$, where s is a basic term. Then we have $\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID} \vdash x = \underline{\underline{\delta}} \cdot s = \underline{\underline{\delta}}$ and, using (i), $\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID} \vdash x + y = y$.

(iv). $x \equiv \underline{\underline{a}} \cdot s$, where $a \in A$ and s is a basic term. From the deduction rules we obtain $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x \xrightarrow{a} s$ and $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models x + y \xrightarrow{a} s$. Since $x + y \sim_{\mathrm{BPA}^-_{\mathrm{drt}}\text{-ID}} y$, we then also have $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \models y \xrightarrow{a} t$ for some t such that $s \sim_{\mathrm{BPA}^-_{\mathrm{drt}}\text{-ID}} t$. By the induction hypothesis we have $\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID} \vdash s = t$. From Lemma 2.5.16(ii) we have $\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID} \vdash y = \underline{\underline{a}} \cdot t + y$. So, $\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID} \vdash x + y = \underline{\underline{a}} \cdot s + y = \underline{\underline{a}} \cdot t + y = y$.

(v). $x \equiv s + t$, where s and t are basic terms. Since $s + t + y \sim_{\text{BPA}^-_{\text{drt}}\text{-ID}} y$, we also have $s + y \sim_{\text{BPA}^-_{\text{drt}}\text{-ID}} y$ and $t + y \sim_{\text{BPA}^-_{\text{drt}}\text{-ID}} y$. By the induction hypothesis we then have $\text{BPA}^-_{\text{drt}}\text{–ID} \vdash s + y = y, t + y = y$. So, $\text{BPA}^-_{\text{drt}}\text{–ID} \vdash x + y = s + t + y = s + y = y$.

(vi). $x \equiv \sigma_{\text{rel}}(s)$, where s is a basic term. From the deduction rules we now have that $T(\text{BPA}^-_{\text{drt}}\text{–ID}) \vDash \sigma_{\text{rel}}(s) \xrightarrow{\sigma} s$ and since $x + y \sim_{\text{BPA}^-_{\text{drt}}\text{-ID}} y$ we also have $T(\text{BPA}^-_{\text{drt}}\text{–ID}) \vDash y \xrightarrow{\sigma} t, x + y \xrightarrow{\sigma} s + t$ for some t such that $s + t \sim_{\text{BPA}^-_{\text{drt}}\text{-ID}} t$. By Lemma 2.5.16(v) we have $\text{BPA}^-_{\text{drt}}\text{–ID} \vdash y = \sigma_{\text{rel}}(t) + \nu_{\text{rel}}(y)$. By the induction hypothesis we have $\text{BPA}^-_{\text{drt}}\text{–ID} \vdash s + t = t$. So, $\text{BPA}^-_{\text{drt}}\text{–ID} \vdash x + y = \sigma_{\text{rel}}(s) + y = \sigma_{\text{rel}}(s) + \sigma_{\text{rel}}(t) + \nu_{\text{rel}}(y) = \sigma_{\text{rel}}(s + t) + \nu_{\text{rel}}(y) = \sigma_{\text{rel}}(t) + \nu_{\text{rel}}(y) = y$.

■

Remark 2.5.18 (Completeness of $\text{BPA}^-_{\text{drt}}$–ID)
Completeness of $\text{BPA}^-_{\text{drt}}$–ID is also claimed (without proof) in Section 2.12.1 of [13] (where $\text{BPA}^-_{\text{drt}}$–ID is called $\text{BPA}_{\delta\text{dt}}$), and in Theorem 2.2 of [11].

2.6 Soundness and Completeness of $\text{BPA}^+_{\text{drt}}$–ID

Definition 2.6.1 (Signature of BPA_{drt}–ID)
The signature of BPA_{drt}–ID consists of the *undelayable atomic actions* $\{\underline{\underline{a}} | a \in A\}$, the *delayable atomic actions* $\{a | a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *delayable deadlock constant* δ, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *time unit delay operator* σ_{rel}, the *"now" operator* ν_{rel}, and the *unbounded start delay operator* $\lfloor\ \rfloor^{\omega}$.

Definition 2.6.2 (Axioms of BPA_{drt}–ID)
The process algebra BPA_{drt}–ID is axiomatized by the axioms of $\text{BPA}^-_{\text{drt}}$–ID given in Definition 2.5.2 on page 22, and Axioms ATS and USD shown in Table 14: BPA_{drt}–ID = A1–A5 + DRT1–DRT5 + DCS1–DCS4 + ATS + USD.

$a = \lfloor \underline{\underline{a}} \rfloor^{\omega}$	ATS
$\lfloor x \rfloor^{\omega} = \nu_{\text{rel}}(x) + \sigma_{\text{rel}}(\lfloor x \rfloor^{\omega})$	USD

Table 14: Axioms for delayable actions.

Definition 2.6.3 (Recursion Principle for BPA_{drt}–ID)
Next to the axioms mentioned in Definition 2.6.2, the system $\text{BPA}^+_{\text{drt}}$–ID also contains the *recursion principle* RSP(USD) shown in Table 15 on the following page. For more information on recursion principles and their status with respect to axioms, see [14].

Remark 2.6.4 (Notation $\text{BPA}^+_{\text{drt}}$–ID)
By superscripting a theory with a "+" (e.g. $\text{BPA}^+_{\text{drt}}$–ID), we indicate the presence of the recursion principle RSP(USD). Note that in [9, 11] the notation BPA_{drt}–ID + RSP(USD) is used instead of $\text{BPA}^+_{\text{drt}}$–ID. However, we find that notation cumbersome, as it clutters up the formulae.

$$y = \nu_{\mathrm{rel}}(x) + \sigma_{\mathrm{rel}}(y) \quad \Longrightarrow \quad y = \lfloor x \rfloor^{\omega} \qquad \text{RSP(USD)}$$

Table 15: Recursive Specification Principle for the Unbounded Start Delay.

Definition 2.6.5 (Semantics of $\mathrm{BPA_{drt}}$–ID)
The semantics of $\mathrm{BPA_{drt}}$–ID are given by the term deduction system $T(\mathrm{BPA_{drt}}\text{–ID})$ induced by the deduction rules for $\mathrm{BPA^{-}_{drt}}$–ID given in Definition 2.5.4 on page 23, and the deduction rules for delayable actions given in Table 16.

$$a \xrightarrow{a} \surd \qquad a \xrightarrow{\sigma} a \qquad \delta \xrightarrow{\sigma} \delta$$

$$\frac{x \xrightarrow{a} x'}{\lfloor x \rfloor^{\omega} \xrightarrow{a} x'} \qquad \frac{x \xrightarrow{a} \surd}{\lfloor x \rfloor^{\omega} \xrightarrow{a} \surd} \qquad \lfloor x \rfloor^{\omega} \xrightarrow{\sigma} \lfloor x \rfloor^{\omega}$$

Table 16: Deduction rules for delayable actions.

Definition 2.6.6 (Bisimulation and Bisimulation Model for $\mathrm{BPA_{drt}}$–ID)
Bisimulation for $\mathrm{BPA_{drt}}$–ID and the corresponding bisimulation model are defined in the same way as for $\mathrm{BPA^{-}_{drt}}$–δ and BPA respectively. Replace "$\mathrm{BPA^{-}_{drt}}$–δ" by "$\mathrm{BPA_{drt}}$–ID" in Definition 2.4.5 on page 14 and "BPA" by "$\mathrm{BPA_{drt}}$–ID" in Definition 2.2.11 on page 8.

Definition 2.6.7 (Basic Terms of $\mathrm{BPA_{drt}}$–ID)
We define $(\sigma, \underline{\underline{\delta}}, \delta)$*-basic terms* inductively as follows:

(i). If $a \in A_\delta$, then $\underline{\underline{a}}$ and a are $(\sigma, \underline{\underline{\delta}}, \delta)$-basic terms,

(ii). if $a \in A_\delta$ and t is a $(\sigma, \underline{\underline{\delta}}, \delta)$-basic term, then $\underline{\underline{a}} \cdot t$ and $a \cdot t$ are $(\sigma, \underline{\underline{\delta}}, \delta)$-basic terms,

(iii). if t and s are $(\sigma, \underline{\underline{\delta}}, \delta)$-basic terms, then $t + s$ is a $(\sigma, \underline{\underline{\delta}}, \delta)$-basic term,

(iv). if t is a $(\sigma, \underline{\underline{\delta}}, \delta)$-basic term, then $\sigma_{\mathrm{rel}}(t)$ is a $(\sigma, \underline{\underline{\delta}}, \delta)$-basic term.

From now on, if we speak of basic terms in the context of $\mathrm{BPA_{drt}}$–ID, we mean $(\sigma, \underline{\underline{\delta}}, \delta)$-basic terms.

Definition 2.6.8 (Number of Symbols of a $\mathrm{BPA_{drt}}$–ID term)
We define $n(x)$, the number of symbols of x, is inductively as follows:

(i). For $a \in A_\delta$, we define $n(\underline{\underline{a}}) = n(a) = 1$,

(ii). for closed $\mathrm{BPA_{drt}}$–ID terms x and y, we define $n(x + y) = n(x \cdot y) = n(x) + n(y) + 1$,

(iii). for a closed $\mathrm{BPA_{drt}}$–ID term x, we define $n(\sigma_{\mathrm{rel}}(x)) = n(\nu_{\mathrm{rel}}(x)) = n(\lfloor x \rfloor^{\omega}) = n(x) + 1$.

Proposition 2.6.9 (Properties of $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID, Part I)
For BPA_{drt}–ID terms x and y, and any $a \in A_\delta$, we have the following equalities:

(i). $BPA^{+}_{drt}\text{–}ID \vdash \lfloor a \rfloor^{\omega} = a$

(ii). $BPA^{+}_{drt}\text{–}ID \vdash \lfloor x \cdot y \rfloor^{\omega} = \lfloor x \rfloor^{\omega} \cdot y$

(iii). $BPA^{+}_{drt}\text{–}ID \vdash \lfloor x + y \rfloor^{\omega} = \lfloor x \rfloor^{\omega} + \lfloor y \rfloor^{\omega}$

(iv). $BPA^{+}_{drt}\text{–}ID \vdash \lfloor \sigma_{rel}(x) \rfloor^{\omega} = \delta$

(v). $BPA_{drt}\text{–}ID \vdash \nu_{rel}(a) = \underline{\underline{a}}$

Proof

(i). Consider the following computation:

$$\begin{aligned}
\mathrm{BPA}_{\mathrm{drt}}\text{–ID} \vdash a &= \lfloor \underline{\underline{a}} \rfloor^{\omega} \\
&= \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) \\
&= \underline{\underline{a}} + \sigma_{\mathrm{rel}}(a) \\
&= \underline{\underline{a}} + \underline{\underline{\delta}} + \sigma_{\mathrm{rel}}(a) \\
&= \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \nu_{\mathrm{rel}}\left(\sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) + \sigma_{\mathrm{rel}}(a) \\
&= \nu_{\mathrm{rel}}\left(\underline{\underline{a}} + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) + \sigma_{\mathrm{rel}}(a) \\
&= \nu_{\mathrm{rel}}\left(\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) + \sigma_{\mathrm{rel}}(a) \\
&= \nu_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) + \sigma_{\mathrm{rel}}(a) \\
&= \nu_{\mathrm{rel}}(a) + \sigma_{\mathrm{rel}}(a)
\end{aligned}$$

Using RSP(USD), we obtain:

$$\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash a = \lfloor a \rfloor^{\omega}$$

(ii). Consider the following computation:

$$\begin{aligned}
\mathrm{BPA}_{\mathrm{drt}}\text{–ID} \vdash \lfloor x \rfloor^{\omega} \cdot y &= \left(\nu_{\mathrm{rel}}(x) + \sigma_{\mathrm{rel}}(\lfloor x \rfloor^{\omega})\right) \cdot y \\
&= \nu_{\mathrm{rel}}(x) \cdot y + \sigma_{\mathrm{rel}}(\lfloor x \rfloor^{\omega}) \cdot y \\
&= \nu_{\mathrm{rel}}(x \cdot y) + \sigma_{\mathrm{rel}}(\lfloor x \rfloor^{\omega} \cdot y)
\end{aligned}$$

Using RSP(USD) (with x instantiated by $x \cdot y$ and y by $\lfloor x \rfloor^{\omega} \cdot y$), we obtain:

$$\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash \lfloor x \rfloor^{\omega} \cdot y = \lfloor x \cdot y \rfloor^{\omega}$$

(iii). Consider the following computation:

$$\begin{aligned}
\mathrm{BPA}_{\mathrm{drt}}\text{–ID} \vdash \lfloor x \rfloor^{\omega} + \lfloor y \rfloor^{\omega} &= \nu_{\mathrm{rel}}(x) + \sigma_{\mathrm{rel}}(\lfloor x \rfloor^{\omega}) + \nu_{\mathrm{rel}}(y) + \sigma_{\mathrm{rel}}(\lfloor y \rfloor^{\omega}) \\
&= \nu_{\mathrm{rel}}(x + y) + \sigma_{\mathrm{rel}}(\lfloor x \rfloor^{\omega} + \lfloor y \rfloor^{\omega})
\end{aligned}$$

Using RSP(USD), we obtain:

$$\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash \lfloor x \rfloor^{\omega} + \lfloor y \rfloor^{\omega} = \lfloor x + y \rfloor^{\omega}$$

(iv). Consider the following computation:

$$\begin{aligned} \mathrm{BPA}_{\mathrm{drt}}\text{–ID} \vdash \delta &= \lfloor \underline{\underline{\delta}} \rfloor^{\omega} \\ &= \nu_{\mathrm{rel}}(\underline{\underline{\delta}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{\delta}} \rfloor^{\omega}\right) \\ &= \underline{\underline{\delta}} + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{\delta}} \rfloor^{\omega}\right) \\ &= \sigma_{\mathrm{rel}}(\delta) \\ &= \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x)) + \sigma_{\mathrm{rel}}(\delta) \end{aligned}$$

Using RSP(USD), we obtain:

$$\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash \delta = \lfloor \sigma_{\mathrm{rel}}(x) \rfloor^{\omega}$$

(v). Consider the following computation:

$$\begin{aligned} \mathrm{BPA}_{\mathrm{drt}}\text{–ID} \vdash \nu_{\mathrm{rel}}(a) &= \nu_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega}) \\ &= \nu_{\mathrm{rel}}\left(\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) \\ &= \nu_{\mathrm{rel}}(\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \\ &= \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(a)) \\ &= \underline{\underline{a}} + \underline{\underline{\delta}} \\ &= \underline{\underline{a}} \end{aligned}$$

So we obtain:

$$\mathrm{BPA}_{\mathrm{drt}}\text{–ID} \vdash \nu_{\mathrm{rel}}(a) = \underline{\underline{a}}$$

■

Remark 2.6.10 (Properties of $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID, Part I)
The equalities of Proposition 2.6.9 on the page before are not new, but have been described before. See for example [20].

Proposition 2.6.11 (Properties of $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID, Part II)
For any BPA_{drt}-ID term x we have the following equality:

$$BPA^{+}_{drt}\text{-}ID \vdash \delta \cdot x = \delta$$

Proof Using Proposition 2.6.9(ii) we derive:

$$\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash \delta \cdot x = \lfloor \underline{\underline{\delta}} \rfloor^{\omega} \cdot x = \lfloor \underline{\underline{\delta}} \cdot x \rfloor^{\omega} = \lfloor \underline{\underline{\delta}} \rfloor^{\omega} = \delta$$

■

Theorem 2.6.12 (Elimination for $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID)
Let t be a closed BPA_{drt}-ID term. Then there is a basic term s such that BPA^{+}_{drt}-ID $\vdash s = t$.

Proof First a term rewriting system is given. Then, it is shown that this term rewriting system is strongly normalizing and that the normal forms of the closed $\mathrm{BPA}_{\mathrm{drt}}$–ID terms are basic terms.

The term rewriting system is given in Table 17 on the following page. The rewriting rules RA4, RA5, RDRT2, RATS, and RDCS1–RDCS4 are obtained directly from the axioms.

$(x+y)\cdot z \to x\cdot z + y\cdot z$	RA4
$(x\cdot y)\cdot z \to x\cdot(y\cdot z)$	RA5
$\sigma_{\mathrm{rel}}(x)\cdot y \to \sigma_{\mathrm{rel}}(x\cdot y)$	RDRT2
$\lfloor \underline{\underline{a}} \rfloor^{\omega} \to a$	RATS
$\lfloor a \rfloor^{\omega} \to a$	RUSD1
$\lfloor x\cdot y \rfloor^{\omega} \to \lfloor x \rfloor^{\omega}\cdot y$	RUSD2
$\lfloor x+y \rfloor^{\omega} \to \lfloor x \rfloor^{\omega} + \lfloor y \rfloor^{\omega}$	RUSD3
$\lfloor \sigma_{\mathrm{rel}}(x) \rfloor^{\omega} \to \delta$	RUSD4
$\nu_{\mathrm{rel}}(\underline{\underline{a}}) \to \underline{\underline{a}}$	RDCS1
$\nu_{\mathrm{rel}}(x+y) \to \nu_{\mathrm{rel}}(x) + \nu_{\mathrm{rel}}(y)$	RDCS2
$\nu_{\mathrm{rel}}(x\cdot y) \to \nu_{\mathrm{rel}}(x)\cdot y$	RDCS3
$\nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x)) \to \underline{\underline{\delta}}$	RDCS4
$\nu_{\mathrm{rel}}(a) \to \underline{\underline{a}}$	RDCS5

Table 17: Term rewriting system for $\mathrm{BPA_{drt}}$–ID.

The rewriting rules RUSD1–RUSD4 and RDCS5 are added to deal properly with the recursive definition of ultimate start delay; the corresponding equalities are derivable from the axioms as is shown in Proposition 2.6.9. With the method of the lexicographical path ordering it is shown that the term rewriting system is strongly normalizing. Thereto the operator $\cdot$ is assigned the lexicographical status for the first argument and the following well-founded partial ordering on the signature of $\mathrm{BPA_{drt}}$–ID is defined:

$$\begin{array}{ccccc}
 & & \lfloor\ \rfloor^{\omega} & > & a \\
 & & \vee & & \\
\nu_{\mathrm{rel}} & > & \cdot & > & + \\
\vee & & \vee & & \\
\underline{\underline{a}} & & \sigma_{\mathrm{rel}} & &
\end{array}$$

Now we show that the left-hand side of every rewriting rule is bigger than the right-hand side with respect to the ordering $\succ_{\mathrm{lpo}}$. This is done by the following reductions:

$$\begin{aligned}
\lfloor \underline{\underline{a}} \rfloor^{\omega} &\succ_{\mathrm{lpo}} \lfloor \underline{\underline{a}} \rfloor^{\omega\star} \\
&\succ_{\mathrm{lpo}} a \\
\lfloor a \rfloor^{\omega} &\succ_{\mathrm{lpo}} \lfloor a \rfloor^{\omega\star} \\
&\succ_{\mathrm{lpo}} a \\
\lfloor x\cdot y \rfloor^{\omega} &\succ_{\mathrm{lpo}} \lfloor x\cdot y \rfloor^{\omega\star} \succ_{\mathrm{lpo}} \lfloor x\cdot y \rfloor^{\omega\star}\cdot\lfloor x\cdot y \rfloor^{\omega\star} \succ_{\mathrm{lpo}} \lfloor x \cdot^{\star} y \rfloor^{\omega}\cdot(x\cdot y) \\
&\succ_{\mathrm{lpo}} \lfloor x \rfloor^{\omega}\cdot(x\cdot y) \succ_{\mathrm{lpo}} \lfloor x \rfloor^{\omega}\cdot(x\cdot^{\star} y) \succ_{\mathrm{lpo}} \lfloor x \rfloor^{\omega}\cdot y \\
\lfloor x+y \rfloor^{\omega} &\succ_{\mathrm{lpo}} \lfloor x+y \rfloor^{\omega\star} \succ_{\mathrm{lpo}} \lfloor x+y \rfloor^{\omega\star} + \lfloor x+y \rfloor^{\omega\star} \succ_{\mathrm{lpo}} \lfloor x +^{\star} y \rfloor^{\omega} + \lfloor x +^{\star} y \rfloor^{\omega} \\
&\succ_{\mathrm{lpo}} \lfloor x \rfloor^{\omega} + \lfloor y \rfloor^{\omega}
\end{aligned}$$

$$
\begin{aligned}
\lfloor \sigma_{\mathrm{rel}}(x) \rfloor^{\omega} &\succ_{\mathrm{lpo}} \lfloor \sigma_{\mathrm{rel}}(x) \rfloor^{\omega\star} \\
&\succ_{\mathrm{lpo}} \delta \\
\nu_{\mathrm{rel}}(a) &\succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}{}^{\star}(a) \\
&\succ_{\mathrm{lpo}} \underline{\underline{a}}
\end{aligned}
$$

Note that we do not give reductions for RA4, RA5, RDRT2, and RDCS1–RDCS4 as these already have been given in the proofs of previous elimination theorems, and since the new ordering is a proper extension of the old ones, these proofs remain valid.

It remains to prove that every normal form of a closed $\mathrm{BPA}_{\mathrm{drt}}$–ID term is a basic term. Suppose that s is the normal form of a closed $\mathrm{BPA}_{\mathrm{drt}}$–ID term. Furthermore, suppose that s is not a basic term and that s' is the smallest subterm of s which is not a basic term. We distinguish all possible cases:

(i). s' is of the form $\underline{\underline{a}}$ or a for some $a \in A_\delta$. Then s' is clearly a basic term, so this case does not occur.

(ii). s' is of the form $s_1 \cdot s_2$ for basic terms s_1 and s_2. With respect to basic term s_1 the following cases can be distinguished:

  (a) $s_1 \equiv \underline{\underline{a}}$ for some $a \in A_\delta$. Then s' is a basic term. This contradicts the assumption that s' is not a basic term.

  (b) $s_1 \equiv a$ for some $a \in A_\delta$. Then s' is a basic term, and we have again a contradiction.

  (c) $s_1 \equiv \underline{\underline{a}} \cdot s_1'$ for some $a \in A_\delta$ and basic term s_1'. Then rewriting rule RA5 is applicable, so s' is not a normal form.

  (d) $s_1 \equiv a \cdot s_1'$ for some $a \in A_\delta$ and basic term s_1'. Then rewriting rule RA5 is applicable, so s' is not a normal form.

  (e) $s_1 \equiv s_1' + s_1''$ for some basic terms s_1' and s_2''. Then rewriting rule RA4 is applicable, so s' is not a normal form.

  (f) $s_1 \equiv \sigma_{\mathrm{rel}}(s_1')$ for some basic term s_1'. Then rewriting rule RDRT2 is applicable, so s' is not a normal form.

(iii). s' is of the form $s_1' + s_2'$ for basic terms s_1' and s_2'. Then s' is a basic term itself, so this case cannot happen.

(iv). s' is of the form $\sigma_{\mathrm{rel}}(s'')$ for some basic term s''. Then again s' is a basic term itself, so this case cannot happen either.

(v). s' is of the form $\nu_{\mathrm{rel}}(s'')$, where s'' is a basic term. Then one of RDCS1–RDCS5 can be applied, so s' is not a normal form.

(vi). s' is of the form $\lfloor s'' \rfloor^{\omega}$ for some basic term s''. Then one of RATS or RUSD1–RUSD4 can be applied, so s' is not a normal form.

In every case s' is a basic term or a rewriting rule is applicable. If s' is a basic term this contradicts the assumption that it is not. If a rewriting rule is applicable then s' and s are not a normal form. This contradicts the assumption that s is a normal form. From this contradiction we conclude that s is a basic term. ■

Remark 2.6.13 (Elimination for $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID)
Elimination for $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID is also claimed (without proof) in Theorem 5.3 of [11] (where $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID is called $\mathrm{BPA}_{\mathrm{drt}}$–ID).

Theorem 2.6.14 (Soundness of $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID)
The set of closed BPA_{drt}–ID terms modulo bisimulation equivalence is a model of BPA^{+}_{drt}–ID.

Proof Note that the soundness proofs for Axioms A1–A5, DRT1–DRT5, and DCS1–DCS4 given in the previous sections also remain valid in the setting with delayable actions. This is due to the fact that the underlying model (namely: finite transition systems with σ's) has not changed. Or, stated more concretely: all σ-transitions were considered without regard for whether they resulted from a σ_{rel} operator or otherwise, so the extra σ-transitions introduced by the delayable actions do not matter.

Axiom ATS Take the relation:

$$R = \{(a, \lfloor \underline{\underline{a}} \rfloor^{\omega})\}$$

There are only two transitions possible on the left-hand side of the axiom: $a \xrightarrow{a} \surd$ and $a \xrightarrow{\sigma} a$. The right-hand side can also perform two transitions: $\lfloor \underline{\underline{a}} \rfloor^{\omega} \xrightarrow{a} \surd$ and $\lfloor \underline{\underline{a}} \rfloor^{\omega} \xrightarrow{\sigma} \lfloor \underline{\underline{a}} \rfloor^{\omega}$, and note that $(a, \lfloor \underline{\underline{a}} \rfloor^{\omega}) \in R$.

Axiom USD Take the relation:

$$R = \{(s,s), (\lfloor s \rfloor^{\omega}, \nu_{\mathrm{rel}}(s) + \sigma_{\mathrm{rel}}(\lfloor s \rfloor^{\omega})) \,|\, s \in C(\mathrm{BPA}_{\mathrm{drt}}\text{–ID})\}$$

First we look at the transitions of the left-hand side:

(i). Suppose $\lfloor s \rfloor^{\omega} \xrightarrow{a} p$. Then this must be due to $s \xrightarrow{a} p$. But then also $\nu_{\mathrm{rel}}(s) \xrightarrow{a} p$ and $\nu_{\mathrm{rel}}(s) + \sigma_{\mathrm{rel}}(\lfloor s \rfloor^{\omega}) \xrightarrow{a} p$, and note that $(p,p) \in R$.

(ii). Suppose $\lfloor s \rfloor^{\omega} \xrightarrow{a} \surd$. Then this must be due to $s \xrightarrow{a} \surd$. But then also $\nu_{\mathrm{rel}}(s) \xrightarrow{a} \surd$ and $\nu_{\mathrm{rel}}(s) + \sigma_{\mathrm{rel}}(\lfloor s \rfloor^{\omega}) \xrightarrow{a} \surd$.

(iii). Suppose $\lfloor s \rfloor^{\omega} \xrightarrow{\sigma} p$. Then necessarily $p \equiv \lfloor s \rfloor^{\omega}$. We also have $\sigma_{\mathrm{rel}}(\lfloor s \rfloor^{\omega}) \xrightarrow{\sigma} \lfloor s \rfloor^{\omega}$, hence $\nu_{\mathrm{rel}}(s) + \sigma_{\mathrm{rel}}(\lfloor s \rfloor^{\omega}) \xrightarrow{\sigma} \lfloor s \rfloor^{\omega}$, and note that $(\lfloor s \rfloor^{\omega}, \lfloor s \rfloor^{\omega}) \in R$.

Secondly, we look at the transitions of the right-hand side:

(i). Suppose $\nu_{\mathrm{rel}}(s) + \sigma_{\mathrm{rel}}(\lfloor s \rfloor^{\omega}) \xrightarrow{a} p$. Then this must be due to $s \xrightarrow{a} p$. But then also $\lfloor s \rfloor^{\omega} \xrightarrow{a} p$, and note that $(p,p) \in R$.

(ii). Suppose $\nu_{\mathrm{rel}}(s) + \sigma_{\mathrm{rel}}(\lfloor s \rfloor^{\omega}) \xrightarrow{a} \surd$. Then this must be due to $s \xrightarrow{a} \surd$. But then also $\lfloor s \rfloor^{\omega} \xrightarrow{a} \surd$.

(iii). Suppose $\nu_{\mathrm{rel}}(s) + \sigma_{\mathrm{rel}}(\lfloor s \rfloor^{\omega}) \xrightarrow{\sigma} p$. Then this must be due to $\sigma_{\mathrm{rel}}(\lfloor s \rfloor^{\omega}) \xrightarrow{\sigma} p$ with $p \equiv \lfloor s \rfloor^{\omega}$. Clearly, then also $\lfloor s \rfloor^{\omega} \xrightarrow{\sigma} \lfloor s \rfloor^{\omega}$, and note that $(\lfloor s \rfloor^{\omega}, p) \in R$.

RSP(USD) Suppose that R' is a bisimulation relation between y and $\nu_{\mathrm{rel}}(x) + \sigma_{\mathrm{rel}}(y)$. We must prove that $y \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-ID}} \lfloor x \rfloor^{\omega}$. Then take the symmetric, transitive closure of R' (which is again a bisimulation, see [15]), denoted R'^{ST}, and extend it to a bisimulation relation R between y and $\lfloor x \rfloor^{\omega}$ as follows:

$$R = R'^{\mathrm{ST}} \cup \{(s, \lfloor x \rfloor^{\omega}) \,|\, s \in C(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \wedge y \overset{\sigma}{\Longrightarrow} s\}$$

Now, let s be any closed term such that $y \overset{\sigma}{\Longrightarrow} s$. Then, using induction on the number of σ-transitions, we have that $(y, s) \in R'$.

First, we look at the transitions of the left-hand side, i.e., the transitions of s:

(i). Suppose $s \xrightarrow{a} p$. Then, as $(y, s) \in R'$, $y \xrightarrow{a} q$ such that $(p, q) \in R'$. As $(y, \nu_{\text{rel}}(x) + \sigma_{\text{rel}}(y)) \in R'$, we have $\nu_{\text{rel}}(x) + \sigma_{\text{rel}}(y) \xrightarrow{a} r$ for some r such that $(q, r) \in R'$. Hence, $\nu_{\text{rel}}(x) \xrightarrow{a} r$, hence $x \xrightarrow{a} r$, hence $\lfloor x \rfloor^{\omega} \xrightarrow{a} r$, and note that by transitivity $(p, r) \in R'^{\text{ST}}$, so $(p, r) \in R$.

(ii). Suppose $s \xrightarrow{a} \surd$. Then, as $(y, s) \in R'$, $y \xrightarrow{a} \surd$. As $(y, \nu_{\text{rel}}(x) + \sigma_{\text{rel}}(y)) \in R'$, we have $\nu_{\text{rel}}(x) + \sigma_{\text{rel}}(y) \xrightarrow{a} \surd$. Hence, $\nu_{\text{rel}}(x) \xrightarrow{a} \surd$, hence $x \xrightarrow{a} \surd$, hence $\lfloor x \rfloor^{\omega} \xrightarrow{a} \surd$.

(iii). Suppose $s \xrightarrow{\sigma} p$. Since $y \overset{\sigma}{\Longrightarrow} s$, we have $y \overset{\sigma}{\Longrightarrow} p$. We also have $\lfloor x \rfloor^{\omega} \xrightarrow{\sigma} \lfloor x \rfloor^{\omega}$, and note that $(p, \lfloor x \rfloor^{\omega}) \in R$.

Secondly, we look at the transitions of the right-hand side:

(i). Suppose $\lfloor x \rfloor^{\omega} \xrightarrow{a} q$. Then $x \xrightarrow{a} q$, hence $\nu_{\text{rel}}(x) \xrightarrow{a} q$, hence $\nu_{\text{rel}}(x) + \sigma_{\text{rel}}(y) \xrightarrow{a} q$. As $(y, \nu_{\text{rel}}(x) + \sigma_{\text{rel}}(y)) \in R'$, we know that $y \xrightarrow{a} p$ for some p such that $(p, q) \in R'$. Since $(y, s) \in R'$, we also have $s \xrightarrow{a} r$ for some r such that $(p, r) \in R'$, and note that by symmetry and transitivity $(r, q) \in R'^{\text{ST}}$, so $(r, q) \in R$.

(ii). Suppose $\lfloor x \rfloor^{\omega} \xrightarrow{a} \surd$. Then $x \xrightarrow{a} \surd$, hence $\nu_{\text{rel}}(x) \xrightarrow{a} \surd$, hence $\nu_{\text{rel}}(x) + \sigma_{\text{rel}}(y) \xrightarrow{a} \surd$. As $(y, \nu_{\text{rel}}(x) + \sigma_{\text{rel}}(y)) \in R'$, we know that $y \xrightarrow{a} \surd$. Since $(y, s) \in R'$, we also have $s \xrightarrow{a} \surd$.

(iii). Suppose $\lfloor x \rfloor^{\omega} \xrightarrow{\sigma} q$. Then, it must be the case that $q \equiv \lfloor x \rfloor^{\omega}$. We also have $\sigma_{\text{rel}}(y) \xrightarrow{\sigma} y$, hence $\nu_{\text{rel}}(x) + \sigma_{\text{rel}}(y) \xrightarrow{\sigma} y$, and as $(y, \nu_{\text{rel}}(x) + \sigma_{\text{rel}}(y)) \in R'$, we know that $y \xrightarrow{\sigma} p$ such that $(p, y) \in R'$. As $(y, s) \in R'$, necessarily $s \xrightarrow{\sigma} r$ for some r such that $(p, r) \in R'$. Since $y \overset{\sigma}{\Longrightarrow} s$, we have $y \overset{\sigma}{\Longrightarrow} r$, and note that $(r, \lfloor x \rfloor^{\omega}) \in R$.

■

Remark 2.6.15 (Soundness of $\text{BPA}^{+}_{\text{drt}}$–ID)
Soundness of $\text{BPA}^{+}_{\text{drt}}$–ID is also claimed (without proof) in Theorem 5.4 of [11] (where $\text{BPA}^{+}_{\text{drt}}$–ID is called BPA_{drt}–ID).

Lemma 2.6.16 (Towards Completeness of $\text{BPA}^{+}_{\text{drt}}$–ID)
Let x and y be closed BPA_{drt}–ID terms and let $a \in A$. Then we have:

(i). $T(BPA_{drt}\text{-}ID) \models x \xrightarrow{a} \surd \implies BPA^{+}_{drt}\text{-}ID \vdash x = \underline{\underline{a}} + x,$

(ii). $T(BPA_{drt}\text{-}ID) \models x \xrightarrow{a} y \implies BPA^{+}_{drt}\text{-}ID \vdash x = \underline{\underline{a}} \cdot y + x,$

(iii). $T(BPA_{drt}\text{-}ID) \models x \overset{\sigma}{\nrightarrow} \implies BPA^{+}_{drt}\text{-}ID \vdash x = \nu_{rel}(x),$

(iv). $BPA^{+}_{drt}\text{-}ID \vdash x + \underline{\underline{\delta}} = x,$

(v). $T(BPA_{drt}\text{-}ID) \models x \xrightarrow{\sigma} y \implies BPA^{+}_{drt}\text{-}ID \vdash x = \sigma_{rel}(y) + \nu_{rel}(x),$

(vi). $T(BPA_{drt}\text{-}ID) \models x \xrightarrow{\sigma} x \implies BPA^{+}_{drt}\text{-}ID \vdash x = \lfloor x \rfloor^{\omega},$

(vii). $T(BPA_{drt}\text{-}ID) \models x \xrightarrow{a} y \implies n(x) > n(y),$

(viii). $T(BPA_{drt}\text{-}ID) \models x \xrightarrow{\sigma} y \implies x \equiv y \vee n(x) > n(y)$.

Proof For part (i)–(vi) we assume, by Theorem 2.6.12 and Theorem 2.6.14, without loss of generality, that x is a basic term, and then apply induction on the structure of basic terms. For part (vii) and (viii) we again have to use induction on the general structure of terms.

(i). Suppose that $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} \surd$. Case 1: $x \equiv \underline{\underline{b}}$, where $b \in A_\delta$. Because $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} \surd$, it must be the case that $b \equiv a$. So we have $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x = \underline{\underline{b}} = \underline{\underline{b}} + \underline{\underline{b}} = \underline{\underline{a}} + \underline{\underline{b}} = \underline{\underline{a}} + x$. Case 2: $x \equiv b$, where $b \in A_\delta$. Because $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} \surd$, it must be the case that $b \equiv a$. So we have $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x = b = \lfloor \underline{\underline{b}} \rfloor^{\omega} = \nu_{\mathrm{rel}}(\underline{\underline{b}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{b}} \rfloor^{\omega}) = \nu_{\mathrm{rel}}(\underline{\underline{b}}) + \nu_{\mathrm{rel}}(\underline{\underline{b}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{b}} \rfloor^{\omega}) = \underline{\underline{b}} + \lfloor \underline{\underline{b}} \rfloor^{\omega} = \underline{\underline{a}} + b = \underline{\underline{a}} + x$. Case 3: $x \equiv \underline{\underline{b}} \cdot x'$, where $b \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} \surd$, so this case does not occur. Case 4: $x \equiv b \cdot x'$, where $b \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} \surd$, so this case does not occur. Case 5: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} \surd$, necessarily $T(\mathrm{BPA_{drt}\text{-}ID}) \models x' \xrightarrow{a} \surd$ or $T(\mathrm{BPA_{drt}\text{-}ID}) \models x'' \xrightarrow{a} \surd$. Therefore, by the induction hypothesis, $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x' = \underline{\underline{a}} + x'$ or $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x'' = \underline{\underline{a}} + x''$. But then in both cases $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x = x' + x'' = \underline{\underline{a}} + x' + x'' = \underline{\underline{a}} + x$. Case 6: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} \surd$, so this case does not occur.

(ii). Suppose that $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} y$. Case 1: $x \equiv \underline{\underline{b}}$, where $b \in A_\delta$. This is in contradiction with $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} y$, so this case does not occur. Case 2: $x \equiv b$, where $b \in A_\delta$. This is in contradiction with $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} y$, so this case does not occur. Case 3: $x \equiv \underline{\underline{b}} \cdot x'$, where $b \in A_\delta$ and x' is a basic term. Then, because $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} y$, it must be that $b \equiv a$ and $x' \equiv y$. So, $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x = x + x = \underline{\underline{b}} \cdot x' + x = \underline{\underline{a}} \cdot y + x$. Case 4: $x \equiv b \cdot x'$, where $b \in A_\delta$ and x' is a basic term. Then, because $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} y$, it must be that $b \equiv a$ and $x' \equiv y$. So, $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x = b \cdot x' = a \cdot y = (\underline{\underline{a}} + a) \cdot y = \underline{\underline{a}} \cdot y + a \cdot y = \underline{\underline{a}} \cdot y + x$. Case 5: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} y$, necessarily $T(\mathrm{BPA_{drt}\text{-}ID}) \models x' \xrightarrow{a} y$ or $T(\mathrm{BPA_{drt}\text{-}ID}) \models x'' \xrightarrow{a} y$. Therefore, by the induction hypothesis, $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x' = \underline{\underline{a}} \cdot y + x'$ or $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x'' = \underline{\underline{a}} \cdot y + x''$. But then in both cases $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x = x' + x'' = \underline{\underline{a}} \cdot y + x' + x'' = \underline{\underline{a}} \cdot y + x$. Case 6: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \xrightarrow{a} y$, so this case does not occur.

(iii). Suppose that $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \overset{\sigma}{\nrightarrow}$. Case 1: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. We have $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x = \underline{\underline{a}} = \nu_{\mathrm{rel}}(\underline{\underline{a}}) = \nu_{\mathrm{rel}}(x)$. Case 2: $x \equiv a$, where $a \in A_\delta$. This is in contradiction with $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \overset{\sigma}{\nrightarrow}$, so this case does not occur. Case 3: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. We have $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x = \underline{\underline{a}} \cdot x' = \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot x' = \nu_{\mathrm{rel}}(\underline{\underline{a}} \cdot x') = \nu_{\mathrm{rel}}(x)$. Case 4: $x \equiv a \cdot x'$, where $a \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \overset{\sigma}{\nrightarrow}$, so this case does not occur. Case 5: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \overset{\sigma}{\nrightarrow}$, necessarily $T(\mathrm{BPA_{drt}\text{-}ID}) \models x' \overset{\sigma}{\nrightarrow}$ and $T(\mathrm{BPA_{drt}\text{-}ID}) \models x'' \overset{\sigma}{\nrightarrow}$. Therefore, by the induction hypothesis, $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x' = \nu_{\mathrm{rel}}(x')$ and $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x'' = \nu_{\mathrm{rel}}(x'')$. But then also $\mathrm{BPA^{+}_{drt}\text{-}ID} \vdash x = x' + x'' = \nu_{\mathrm{rel}}(x') + \nu_{\mathrm{rel}}(x'') = \nu_{\mathrm{rel}}(x' + x'') = \nu_{\mathrm{rel}}(x)$. Case 6: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\mathrm{BPA_{drt}\text{-}ID}) \models x \overset{\sigma}{\nrightarrow}$, so this case does not occur.

(iv). Case 1: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. This is exactly Axiom DRT4. Case 2: $x \equiv a$, where $a \in A_\delta$. Then we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x + \underline{\underline{\delta}} = a + \underline{\underline{\delta}} = \lfloor \underline{\underline{a}} \rfloor^{\omega} + \underline{\underline{\delta}} = \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) + \underline{\underline{\delta}} = \underline{\underline{a}} + \underline{\underline{\delta}} + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) = \underline{\underline{a}} + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) = \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) = \lfloor \underline{\underline{a}} \rfloor^{\omega} = a = x$. Case 3: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. Then we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x + \underline{\underline{\delta}} = \underline{\underline{a}} \cdot x' + \underline{\underline{\delta}} = \underline{\underline{a}} \cdot x' + \underline{\underline{\delta}} \cdot x' = (\underline{\underline{a}} + \underline{\underline{\delta}}) \cdot x' = \underline{\underline{a}} \cdot x' = x$. Case 4: $x \equiv a \cdot x'$, where $a \in A_\delta$ and x' is a basic term. Then, by case 2, $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x + \underline{\underline{\delta}} = a \cdot x' + \underline{\underline{\delta}} = a \cdot x' + \underline{\underline{\delta}} \cdot x' = (a + \underline{\underline{\delta}}) \cdot x' = a \cdot x' = x$. Case 5: $x \equiv x' + x''$, where x' and x'' are basic terms. By the induction hypothesis we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x' + \underline{\underline{\delta}} = x', x'' + \underline{\underline{\delta}} = x''$. So $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x + \underline{\underline{\delta}} = x' + x'' + \underline{\underline{\delta}} = x' + x'' = x$. Case 6: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is exactly Axiom DRT5.

(v). Suppose that $T(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \vDash x \xrightarrow{\sigma} y$. Case 1: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \vDash x \xrightarrow{\sigma} y$, so this case does not occur. Case 2: $x \equiv a$, where $a \in A_\delta$. Because $T(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \vDash x \xrightarrow{\sigma} y$, it must be the case that $y \equiv a$. So we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x = a = \lfloor \underline{\underline{a}} \rfloor^{\omega} = \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) = \sigma_{\mathrm{rel}}(a) + \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(a) + \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \nu_{\mathrm{rel}}\left(\sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) = \sigma_{\mathrm{rel}}(a) + \nu_{\mathrm{rel}}\left(\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) = \sigma_{\mathrm{rel}}(a) + \nu_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) = \sigma_{\mathrm{rel}}(a) + \nu_{\mathrm{rel}}(a) = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$. Case 3: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \vDash x \xrightarrow{\sigma} y$, so this case does not occur. Case 4: $x \equiv a \cdot x'$, where $a \in A_\delta$ and x' is a basic term. Because $T(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \vDash x \xrightarrow{\sigma} y$, it must be the case that $y \equiv a \cdot x'$. So we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x = a \cdot x' = \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot x' = \left(\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) \cdot x' = \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot x' + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) \cdot x' = \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) \cdot x' + \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot x' + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(a \cdot x') + \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot x' + \underline{\underline{\delta}} \cdot x' = \sigma_{\mathrm{rel}}(y) + (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \underline{\underline{\delta}}) \cdot x' = \sigma_{\mathrm{rel}}(y) + \left(\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \nu_{\mathrm{rel}}\left(\sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right)\right) \cdot x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}\left(\underline{\underline{a}} + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) \cdot x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}\left(\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) \cdot x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) \cdot x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(a) \cdot x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(a \cdot x') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$. Case 5: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \vDash x \xrightarrow{\sigma} y$, necessarily (1) $T(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \vDash x' \xrightarrow{\sigma} y, x'' \stackrel{\sigma}{\nrightarrow}$, or, (2) $T(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \vDash x' \stackrel{\sigma}{\nrightarrow}, x'' \xrightarrow{\sigma} y$, or, (3) $T(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \vDash x' \xrightarrow{\sigma} y', x'' \xrightarrow{\sigma} y''$ where $y \equiv y' + y''$. In the first case, by the induction hypothesis, we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x')$, and, by (iii), $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x'' = \nu_{\mathrm{rel}}(x'')$. Therefore, $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x = x' + x'' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x') + \nu_{\mathrm{rel}}(x'') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x' + x'') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$. The second case is treated analogously. In the third case we have, by the induction hypothesis, $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x' = \sigma_{\mathrm{rel}}(y') + \nu_{\mathrm{rel}}(x'), x'' = \sigma_{\mathrm{rel}}(y'') + \nu_{\mathrm{rel}}(x'')$. Therefore we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x = x' + x'' = \sigma_{\mathrm{rel}}(y') + \nu_{\mathrm{rel}}(x') + \sigma_{\mathrm{rel}}(y'') + \nu_{\mathrm{rel}}(x'') = \sigma_{\mathrm{rel}}(y' + y'') + \nu_{\mathrm{rel}}(x' + x'') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$. Case 6: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. Because $T(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \vDash x \xrightarrow{\sigma} y$, it must be the case that $x' \equiv y$. So we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x = \sigma_{\mathrm{rel}}(x') = \sigma_{\mathrm{rel}}(y) = \sigma_{\mathrm{rel}}(y) + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x')) = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$.

(vi). Suppose that $T(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \vDash x \xrightarrow{\sigma} x$. Case 1: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \vDash x \xrightarrow{\sigma} x$, so this case does not occur. Case 2: $x \equiv a$, where $a \in A_\delta$. Then we have, using Proposition 2.6.9(i), $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x = a = \lfloor a \rfloor^{\omega} = \lfloor x \rfloor^{\omega}$. Case 3: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}\text{–ID}) \vDash x \xrightarrow{\sigma} x$, so this case does not occur. Case 4: $x \equiv a \cdot x'$, where $a \in A_\delta$ and x' is a basic term. Then we can derive, using Proposition 2.6.9(i) and (ii), $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x = a \cdot x' = \lfloor a \rfloor^{\omega} \cdot x' = \lfloor a \cdot x' \rfloor^{\omega} = \lfloor x \rfloor^{\omega}$. Case 5: $x \equiv x' + x''$, where x' and x'' are basic terms. Then we can derive, using Proposition 2.6.9(iii) and the induction hypothesis, $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash x = x' + x'' = \lfloor x \rfloor^{\omega} + \lfloor x' \rfloor^{\omega} = \lfloor x' + x'' \rfloor^{\omega} = \lfloor x \rfloor^{\omega}$. Case 6:

$x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{\sigma} x$, so this case does not occur.

(vii). Suppose that $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{a} y$. Case 1: $x \equiv \underline{\underline{b}}$, where $b \in A_\delta$. This is in contradiction with $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{a} y$, so this case does not occur. Case 2: $x \equiv \underline{b}$, where $b \in A_\delta$. This is in contradiction with $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{a} y$, so this case does not occur. Case 3: x is of the form $x' \cdot x''$, for certain terms x' and x''. Then, by $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{a} y$, we either have $T(\mathrm{BPA_{drt}}\text{–ID}) \models x' \xrightarrow{a} \surd$ and $y \equiv x''$, or we have $T(\mathrm{BPA_{drt}}\text{–ID}) \models x' \xrightarrow{a} x'''$ and $y \equiv x''' \cdot x''$ for a certain term x'''. In the first case, we have $n(x) = n(x' \cdot x'') = n(x') + n(x'') + 1 > n(x'') = n(y)$, and in the second we can apply the induction hypothesis to arrive at $n(x) > n(x''')$, so we get $n(x) = n(x' \cdot x'') = n(x') + n(x'') + 1 > n(x') + n(x''') + 1 = n(x''' \cdot x') = n(y)$. Case 4: $x \equiv x' + x''$, for certain terms x' and x''. Since $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{a} y$, necessarily $T(\mathrm{BPA_{drt}}\text{–ID}) \models x' \xrightarrow{a} y$ or $T(\mathrm{BPA_{drt}}\text{–ID}) \models x'' \xrightarrow{a} y$. Therefore, by the induction hypothesis, $n(x') > n(y)$ or $n(x'') > n(y)$. In both cases $n(x) = n(x' + x'') = n(x') + n(x'') + 1 > n(y)$. Case 5: $x \equiv \sigma_{\mathrm{rel}}(x')$, for a certain term x'. This is in contradiction with $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{a} y$, so this case does not occur. Case 6: $x \equiv \nu_{\mathrm{rel}}(x')$, for a certain term x'. Then, by $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{a} y$, we also have $T(\mathrm{BPA_{drt}}\text{–ID}) \models x' \xrightarrow{a} y$, and using the induction hypothesis, $n(x) = n(\nu_{\mathrm{rel}}(x')) = n(x') + 1 > n(y)$. Case 7: $x \equiv \lfloor x' \rfloor^{\omega}$, for a certain term x'. Then, by $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{a} y$, we also have $T(\mathrm{BPA_{drt}}\text{–ID}) \models x' \xrightarrow{a} y$, and using the induction hypothesis, $n(x) = n(\lfloor x' \rfloor^{\omega}) = n(x') + 1 > n(y)$.

(viii). Suppose that $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$. Case 1: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. This is in contradiction with $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 2: $x \equiv \underline{a}$, where $a \in A_\delta$. As $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, necessarily $y \equiv x$. Case 3: $x \equiv x' \cdot x''$ for certain terms x' and x''. Then, because $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, either $x \equiv y$ and we are done, or $T(\mathrm{BPA_{drt}}\text{–ID}) \models x' \xrightarrow{\sigma} x'''$ and $y \equiv x''' \cdot x''$ for some term $x''' \not\equiv x'$. In that case, we can apply the induction hypothesis to arrive at $n(x') > n(x''')$, so we get $n(x) = n(x' \cdot x'') = n(x') + n(x'') + 1 > n(x''') + n(x'') + 1 = n(x''' \cdot x'') = n(y)$. Case 4: $x \equiv x' + x''$, for certain terms x' and x''. As $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, necessarily (1) $T(\mathrm{BPA_{drt}}\text{–ID}) \models x' \xrightarrow{\sigma} y, x'' \not\xrightarrow{\sigma}$, or, (2) $T(\mathrm{BPA_{drt}}\text{–ID}) \models x' \not\xrightarrow{\sigma}, x'' \xrightarrow{\sigma} y$, or, (3) $T(\mathrm{BPA_{drt}}\text{–ID}) \models x' \xrightarrow{\sigma} y', x'' \xrightarrow{\sigma} y''$ where $y \equiv y' + y''$. In the first case, by the induction hypothesis, $x' \equiv y$ or $n(x') > n(y)$. If $x' \equiv y$ we have $n(x) = n(x' + x'') = n(x') + n(x'') + 1 > n(x') = n(y)$, otherwise we have $n(x) = n(x' + x'') = n(x') + n(x'') + 1 > n(y)$. The second case is treated analogously. In the third case, by the induction hypothesis, $x' \equiv y'$ or $n(x') > n(y')$, and, $x'' \equiv y''$ or $n(x'') > n(y'')$. If $x' \equiv y'$ and $x'' \equiv y''$, we have $x \equiv x' + x'' \equiv y' + y'' \equiv y$, and we are done. Otherwise, we have $n(x') + n(x'') > n(y') + n(y'')$, so $n(x) = n(x' + x'') = n(x') + n(x'') + 1 > n(y') + n(y'') + 1 = n(y)$. Case 5: $x \equiv \sigma_{\mathrm{rel}}(x')$, for a certain term x'. Because $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, it must be the case that $x' \equiv y$. Then we have $n(x) = n(\sigma_{\mathrm{rel}}(x')) = n(x') + 1 = n(y) + 1 > n(y)$. Case 6: $x \equiv \nu_{\mathrm{rel}}(x')$, for a certain term x'. This is in contradiction with $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 7: $x \equiv \lfloor x' \rfloor^{\omega}$, for a certain term x'. Because $T(\mathrm{BPA_{drt}}\text{–ID}) \models x \xrightarrow{\sigma} y$, it must be the case that $x \equiv y$, and we are done.

■

Theorem 2.6.17 (Completeness of $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID)
The axiom system BPA^{+}_{drt}–ID is a complete axiomatization of the set of closed BPA_{drt}–ID terms modulo bisimulation equivalence.

Proof Suppose that $s + t \sim_{\mathrm{BPA_{drt}\text{-}ID}} t$. We will prove that $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash s + t = t$. By Theorem 2.6.12 we can restrict ourselves to basic terms s and t. The proof is done with induction on $n(s)$, using Lemma 2.6.16(vii)–(viii) and case distinction on the form of basic term s.

(i). $s \equiv \underline{\underline{\delta}}$. The equality that we must prove, i.e. $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash \underline{\underline{\delta}} + t = t$, is Lemma 2.6.16(iv).

(ii). $s \equiv \underline{\underline{a}}$, where $a \in A$. Then $s \xrightarrow{a} \surd$. Then also $s + t \xrightarrow{a} \surd$. Since $s + t$ and t are bisimilar also $t \xrightarrow{a} \surd$. By Lemma 2.6.16(i) we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash t = \underline{\underline{a}} + t$. Consider the following computation: $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash s + t = \underline{\underline{a}} + t = t$.

(iii). $s \equiv \delta$. Then $\delta \xrightarrow{\sigma} \delta$. Therefore $s + t \xrightarrow{\sigma} s + t'$ and $t \xrightarrow{\sigma} t'$ with $s + t' \sim_{\mathrm{BPA_{drt}\text{-}ID}} t'$. With Lemma 2.6.16(v) we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash t = \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t)$. Two cases need to be considered:

- (a) $t \equiv t'$. Now, $s + t \xrightarrow{\sigma} s + t$ and $t \xrightarrow{\sigma} t$, so by Lemma 2.6.16(vi) we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash s + t = \lfloor s + t \rfloor^{\omega}$ and $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash t = \lfloor t \rfloor^{\omega}$. So we can derive, using Proposition 2.6.9(iii) and Lemma 2.6.16(iv): $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash s + t = \lfloor s + t \rfloor^{\omega} = \lfloor s \rfloor^{\omega} + \lfloor t \rfloor^{\omega} = \lfloor \delta \rfloor^{\omega} + \lfloor t \rfloor^{\omega} = \delta + \lfloor t \rfloor^{\omega} = \lfloor \underline{\underline{\delta}} \rfloor^{\omega} + \lfloor t \rfloor^{\omega} = \lfloor \underline{\underline{\delta}} + t \rfloor^{\omega} = \lfloor t \rfloor^{\omega} = t$.
- (b) $t \not\equiv t'$. Now, by Lemma 2.6.16(viii), $n(t') < n(t)$. Therefore, the induction hypothesis is applicable: $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash \delta + t' = t'$. Consider the following computation: $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash s + t = \delta + t = \lfloor \underline{\underline{\delta}} \rfloor^{\omega} + t = \nu_{\mathrm{rel}}(\underline{\underline{\delta}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{\delta}} \rfloor^{\omega}\right) + t = \underline{\underline{\delta}} + \sigma_{\mathrm{rel}}(\delta) + t = \sigma_{\mathrm{rel}}(\delta) + t = \sigma_{\mathrm{rel}}(\delta) + \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(\delta + t') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t) = t$.

(iv). $s \equiv a$, where $a \in A$. Then $s \xrightarrow{a} \surd$. Therefore $s + t \xrightarrow{a} \surd$ and, since $s + t \sim_{\mathrm{BPA_{drt}\text{-}ID}} t$, $t \xrightarrow{a} \surd$. Using Lemma 2.6.16(i) we obtain $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash t = \underline{\underline{a}} + t$. We also have $s \xrightarrow{\sigma} s$. Therefore $s + t \xrightarrow{\sigma} s + t'$ and $t \xrightarrow{\sigma} t'$. From Lemma 2.6.16(v) we obtain: $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash t = \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t)$. Two cases can be distinguished:

- (a) $t \equiv t'$. Now, $s + t \xrightarrow{\sigma} s + t$ and $t \xrightarrow{\sigma} t$, so by Lemma 2.6.16(vi) we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash s + t = \lfloor s + t \rfloor^{\omega}$ and $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash t = \lfloor t \rfloor^{\omega}$. So we can derive, using Proposition 2.6.9(iii): $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash s + t = \lfloor s + t \rfloor^{\omega} = \lfloor s \rfloor^{\omega} + \lfloor t \rfloor^{\omega} = \lfloor a \rfloor^{\omega} + \lfloor t \rfloor^{\omega} = a + \lfloor t \rfloor^{\omega} = \lfloor \underline{\underline{a}} \rfloor^{\omega} + \lfloor t \rfloor^{\omega} = \lfloor \underline{\underline{a}} + t \rfloor^{\omega} = \lfloor t \rfloor^{\omega} = t$.
- (b) $t \not\equiv t'$. Now, by Lemma 2.6.16(viii), $n(t') < n(t)$. Therefore, the induction hypothesis is applicable: $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash a + t' = t'$. Consider the following computation: $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash s + t = a + t = \lfloor \underline{\underline{a}} \rfloor^{\omega} + t = \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) + t = \underline{\underline{a}} + \sigma_{\mathrm{rel}}(a) + t = \sigma_{\mathrm{rel}}(a) + t = \sigma_{\mathrm{rel}}(a) + \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(a + t') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t) = t$.

(v). $s \equiv \underline{\underline{\delta}} \cdot s'$, where s' is a basic term. Then we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash s = \underline{\underline{\delta}} \cdot s' = \underline{\underline{\delta}}$ and, using (i), $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash s + t = t$.

(vi). $s \equiv \underline{\underline{a}} \cdot s'$, where $a \in A$ and s' is a basic term. Then $s \xrightarrow{a} s'$ and $s + t \xrightarrow{a} s'$. By Lemma 2.6.16(ii) we have $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID $\vdash t = \underline{\underline{a}} \cdot t' + t$. Since $s + t \sim_{\mathrm{BPA_{drt}\text{-}ID}} t$ we also have $t \xrightarrow{a} t'$ for some t' such that $s' \sim_{\mathrm{BPA_{drt}\text{-}ID}} t'$. By induction we have $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s' = t'$. Consider the following computation: $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s + t = \underline{\underline{a}} \cdot s' + t = \underline{\underline{a}} \cdot t' + t = t$.

(vii). $s \equiv \delta \cdot s'$, where s' is a basic term. Then we have, using Proposition 2.6.11, $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s = \delta \cdot s' = \delta$ and, using (iii), $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s + t = t$.

(viii). $s \equiv a \cdot s'$, where $a \in A$ and s' is a basic term. Then $s \xrightarrow{a} s'$ and $s + t \xrightarrow{a} s'$. Since $s + t \sim_{\mathrm{BPA_{drt}\text{-}ID}} t$ we also have $t \xrightarrow{a} t'$ for some t' such that $s' \sim_{\mathrm{BPA_{drt}\text{-}ID}} t'$. By induction we therefore have $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s' = t'$. We also have $s \xrightarrow{\sigma} s$ and $s + t \xrightarrow{\sigma} s + t''$ and $t \xrightarrow{\sigma} t''$. By Lemma 2.6.16(ii) we have $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash t = \underline{\underline{a}} \cdot t' + t$ and $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash t = \sigma_{\mathrm{rel}}(t'') + \nu_{\mathrm{rel}}(t)$. Two cases can be distinguished:

(a) $t \equiv t''$. Now, $s + t \xrightarrow{\sigma} s + t$ and $t \xrightarrow{\sigma} t$, so by Lemma 2.6.16(vi) we have $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s + t = \lfloor s + t \rfloor^{\omega}$ and $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash t = \lfloor t \rfloor^{\omega}$. So we can derive, using Proposition 2.6.9(i)–(iii): $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s + t = \lfloor s + t \rfloor^{\omega} = \lfloor s \rfloor^{\omega} + \lfloor t \rfloor^{\omega} = \lfloor a \cdot s' \rfloor^{\omega} + \lfloor t \rfloor^{\omega} = \lfloor a \rfloor^{\omega} \cdot s' + \lfloor t \rfloor^{\omega} = a \cdot s' + \lfloor t \rfloor^{\omega} = \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot s' + \lfloor t \rfloor^{\omega} = \lfloor \underline{\underline{a}} \cdot s' \rfloor^{\omega} + \lfloor t \rfloor^{\omega} = \lfloor \underline{\underline{a}} \cdot s' + t \rfloor^{\omega} = \lfloor \underline{\underline{a}} \cdot t' + t \rfloor^{\omega} = \lfloor t \rfloor^{\omega} = t$.

(b) $t \not\equiv t''$. Now, by Lemma 2.6.16(viii), $n(t'') < n(t)$. By the induction hypothesis we then have $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s + t'' = t''$. Consider the following computation: $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s + t = a \cdot s' + t = \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot s' + t = (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega})) \cdot s' + t = (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \cdot s' + t = \underline{\underline{a}} \cdot t' + \sigma_{\mathrm{rel}}(a) \cdot s' + t = \sigma_{\mathrm{rel}}(a) \cdot s' + t = \sigma_{\mathrm{rel}}(a \cdot s') + t = \sigma_{\mathrm{rel}}(s) + t = \sigma_{\mathrm{rel}}(s) + \sigma_{\mathrm{rel}}(t'') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(s + t'') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(t'') + \nu_{\mathrm{rel}}(t) = t$.

(ix). $s \equiv s' + s''$, where s' and s'' are basic terms. Since $s' + s'' + t \sim_{\mathrm{BPA_{drt}\text{-}ID}} t$ we also have $s' + t \sim_{\mathrm{BPA_{drt}\text{-}ID}} t$ and $s'' + t \sim_{\mathrm{BPA_{drt}\text{-}ID}} t$. By induction we have $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s' + t = t$ and $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s'' + t = t$. Then $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s + t = s' + s'' + t = s'' + t = t$.

(x). $s \equiv \sigma_{\mathrm{rel}}(s')$, where s' is a basic term. Then $s \xrightarrow{\sigma} s'$ and $s + t \xrightarrow{\sigma} s' + t'$ and $t \xrightarrow{\sigma} t'$ for some t' such that $s' + t' \sim_{\mathrm{BPA_{drt}\text{-}ID}} t'$. By induction we have $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s' + t' = t'$ and by Lemma 2.6.16(v) we have $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash t = \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t)$. Then $\mathrm{BPA}^{+}_{\mathrm{drt}}\text{–ID} \vdash s + t = \sigma_{\mathrm{rel}}(s') + t = \sigma_{\mathrm{rel}}(s') + \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(s' + t') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t) = t$.

■

Remark 2.6.18 (Completeness of $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID)
Completeness of $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID is also claimed (without proof) in Theorem 5.4 of [11] (where $\mathrm{BPA}^{+}_{\mathrm{drt}}$–ID is called $\mathrm{BPA_{drt}}$–ID).

Definition 2.6.19 (Axioms for the Ultimate Start Delay)
We define Axioms USD1–USD4 for the ultimate start delay as given in Table 18 on the next page. Note that they precisely correspond to the equalities of Proposition 2.6.9(i)–(iv).

Remark 2.6.20 (Proving Soundness and Completeness Indirectly)
Next to proving soundness and completeness directly (outlined in Remarks 2.4.13 on page 17 and 2.2.18 on page 9), we can also take a sound and complete process theory P, and replace some of its axioms by some new axioms that exactly correspond to the equalities of P that are used to prove the completeness of P. The resulting new theory will then also be sound and complete.

$$\lfloor a \rfloor^{\omega} = a \qquad \text{USD1}$$
$$\lfloor x \cdot y \rfloor^{\omega} = \lfloor x \rfloor^{\omega} \cdot y \qquad \text{USD2}$$
$$\lfloor x + y \rfloor^{\omega} = \lfloor x \rfloor^{\omega} + \lfloor y \rfloor^{\omega} \qquad \text{USD3}$$
$$\lfloor \sigma_{\text{rel}}(x) \rfloor^{\omega} = \delta \qquad \text{USD4}$$

Table 18: Additional axioms for the ultimate start delay.

For an example of this method, see Corollaries 2.6.21 and 2.6.22 below. Furthermore, this method is used in the proofs of Corollaries 2.8.25, 2.8.26, 3.6.17, 3.6.18, 3.7.20, 3.7.21, 3.7.26, and 3.7.27.

Corollary 2.6.21 (Soundness of $\mathbf{BPA_{drt}}$–ID + USD1–USD4)
The set of closed BPA_{drt}-ID terms modulo bisimulation equivalence is a model of BPA_{drt}-ID + USD1–USD4.

Proof This follows directly from the soundness of $\text{BPA}^{+}_{\text{drt}}$–ID (see Theorem 2.6.14 on page 38) and the fact that Axioms USD1–USD4 are derivable in $\text{BPA}^{+}_{\text{drt}}$–ID (see Proposition 2.6.9 on page 34). ∎

Corollary 2.6.22 (Completeness of $\mathbf{BPA_{drt}}$–ID + USD1–USD4)
If we add Axioms USD1–USD4 of Table 18 to BPA_{drt}-ID, we again have a complete axiomatization of the set of closed BPA_{drt}-ID terms modulo bisimulation equivalence.

Proof Careful inspection of the dependencies between the proofs in this section reveals that the proof of Theorem 2.6.17 only relies upon RSP(USD) to ensure Proposition 2.6.9(i)–(iv). So, we obviously do not need RSP(USD) anymore if we add the corresponding Axioms USD1–USD4. Note that in this way we get a purely equational axiomatization (i.e. without conditional axioms or principles). ∎

2.7 Soundness and Completeness of $\mathbf{BPA^{-}_{drt}}$

Definition 2.7.1 (Signature of $\mathbf{BPA^{-}_{drt}}$)
The signature of $\text{BPA}^{-}_{\text{drt}}$ consists of the *undelayable atomic actions* $\{\underline{\underline{a}} | a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *immediate deadlock constant* $\dot{\delta}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *time unit delay operator* σ_{rel}, and the *"now" operator* ν_{rel}.

Definition 2.7.2 (Axioms of $\mathbf{BPA^{-}_{drt}}$)
The process algebra $\text{BPA}^{-}_{\text{drt}}$ is axiomatized by the axioms of $\text{BPA}^{-}_{\text{drt}}$–ID given in Definition 2.5.2 on page 22 and Axioms DRTSID, A6ID, A7ID, and DCSID shown in Table 19 on the following page: $\text{BPA}^{-}_{\text{drt}}$ = A1–A5 + A6ID + A7ID + DRT1–DRT5 + DRTSID + DCS1–DCS4 + DCSID.

$$\begin{aligned} \sigma_{\mathrm{rel}}(\dot{\delta}) &= \underline{\underline{\delta}} && \text{DRTSID} \\ x + \dot{\delta} &= x && \text{A6ID} \\ \dot{\delta} \cdot x &= \dot{\delta} && \text{A7ID} \\ \nu_{\mathrm{rel}}(\dot{\delta}) &= \dot{\delta} && \text{DCSID} \end{aligned}$$

Table 19: Axioms for $\dot{\delta}$.

Remark 2.7.3 (A6ID versus A6)
Note that in a setting with immediate deadlock we do not have anymore that $x + \underline{\underline{\delta}} = x$, as $\dot{\delta} + \underline{\underline{\delta}} = \underline{\underline{\delta}}$. We do however still have $\underline{\underline{a}} + \underline{\underline{\delta}} = \underline{\underline{a}}$. See also Remark 2.3.3 on page 12 and Remark 2.5.3 on page 22.

Remark 2.7.4 (Derivability of DRT3 and DRT5)
Note that from Axiom A6ID, DRT1, DRT2, and DRTSID we can derive Axiom DRT3 and DRT5, as we have:

$$\underline{\underline{\delta}} \cdot x = \sigma_{\mathrm{rel}}(\dot{\delta}) \cdot x = \sigma_{\mathrm{rel}}(\dot{\delta} \cdot x) = \sigma_{\mathrm{rel}}(\dot{\delta}) = \underline{\underline{\delta}}$$

and, similarly:

$$\sigma_{\mathrm{rel}}(x) + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(x) + \sigma_{\mathrm{rel}}(\dot{\delta}) = \sigma_{\mathrm{rel}}(x + \dot{\delta}) = \sigma_{\mathrm{rel}}(x)$$

However, we still choose to include DRT3 and DRT5 even for theories that contain $\dot{\delta}$, as our goal is not to find a *minimal* axiomatization, but instead to find a *convenient* one (with regard to ease of proofs and calculations).

Definition 2.7.5 (Summation Convention with Respect to Immediate Deadlock)
In a setting with immediate deadlock, we will use the convention that a summation over the empty set yields the immediate deadlock:

$$\sum_{i \in \emptyset} t_i = \dot{\delta}.$$

Definition 2.7.6 (Semantics of $\mathrm{BPA}^{-}_{\mathrm{drt}}$)
The semantics of $\mathrm{BPA}^{-}_{\mathrm{drt}}$ are given by the term deduction system $T(\mathrm{BPA}^{-}_{\mathrm{drt}})$ induced by the deduction rules for $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID given in Definition 2.5.4 on page 23, *minus* the deduction rule $\sigma_{\mathrm{rel}}(x) \xrightarrow{\sigma} x$, *plus* the deduction rules given in Table 20 on the following page.

Definition 2.7.7 (Bisimulation for $\mathrm{BPA}^{-}_{\mathrm{drt}}$)
Bisimulation for $\mathrm{BPA}^{-}_{\mathrm{drt}}$ is defined as follows; a binary relation R on closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$ terms is a bisimulation if the following transfer conditions hold for all closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$ terms p and q:

(i). If $R^{\mathrm{S}}(p, q)$ and $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models p \xrightarrow{a} p'$, where $a \in A$, then there exists a closed term q', such that $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models q \xrightarrow{a} q'$ and $R^{\mathrm{S}}(p', q')$,

$$\mathrm{ID}(\dot{\delta}) \qquad \frac{\mathrm{ID}(x)}{\mathrm{ID}(x \cdot y)} \qquad \frac{\mathrm{ID}(x),\ \mathrm{ID}(y)}{\mathrm{ID}(x + y)} \qquad \frac{\neg\mathrm{ID}(x)}{\sigma_{\mathrm{rel}}(x) \xrightarrow{\sigma} x}$$

Table 20: Deduction rules for ID.

(ii). If $R^{\mathrm{S}}(p,q)$ and $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models p \xrightarrow{\sigma} p'$, then there exists a closed term q', such that $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models q \xrightarrow{\sigma} q'$ and $R^{\mathrm{S}}(p',q')$,

(iii). If $R^{\mathrm{S}}(p,q)$ and $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models p \xrightarrow{a} \surd$, where $a \in A$, then $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models q \xrightarrow{a} \surd$,

(iv). If $R^{\mathrm{S}}(p,q)$ and $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models \mathrm{ID}(p)$, then $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models \mathrm{ID}(q)$.

Two $\mathrm{BPA}^{-}_{\mathrm{drt}}$ terms p and q are bisimilar, notation $p \sim_{\mathrm{BPA}^{-}_{\mathrm{drt}}} q$, if there exists a bisimulation relation R such that $R(p,q)$.

Definition 2.7.8 (Bisimulation Model for $\mathrm{BPA}^{-}_{\mathrm{drt}}$)
The bisimulation model for $\mathrm{BPA}^{-}_{\mathrm{drt}}$ is defined in the same way as for BPA. Replace "BPA" by "$\mathrm{BPA}^{-}_{\mathrm{drt}}$" in Definition 2.2.11 on page 8.

Definition 2.7.9 (Basic Terms of $\mathrm{BPA}^{-}_{\mathrm{drt}}$)
We define $(\sigma, \underline{\underline{\delta}}, \dot{\delta})$*-basic terms* inductively as follows:

(i). Immediate deadlock $\dot{\delta}$ is a $(\sigma, \underline{\underline{\delta}}, \dot{\delta})$-basic term,

(ii). if $a \in A_{\delta}$, then $\underline{\underline{a}}$ is a $(\sigma, \underline{\underline{\delta}}, \dot{\delta})$-basic term,

(iii). if $a \in A_{\delta}$ and t is a $(\sigma, \underline{\underline{\delta}}, \dot{\delta})$-basic term, then $\underline{\underline{a}} \cdot t$ is a $(\sigma, \underline{\underline{\delta}}, \dot{\delta})$-basic term,

(iv). if t and s are $(\sigma, \underline{\underline{\delta}}, \dot{\delta})$-basic terms, then $t + s$ is a $(\sigma, \underline{\underline{\delta}}, \dot{\delta})$-basic term,

(v). if t is a $(\sigma, \underline{\underline{\delta}}, \dot{\delta})$-basic term, then $\sigma_{\mathrm{rel}}(t)$ is a $(\sigma, \underline{\underline{\delta}}, \dot{\delta})$-basic term.

From now on, if we speak of basic terms in the context of $\mathrm{BPA}^{-}_{\mathrm{drt}}$, we mean $(\sigma, \underline{\underline{\delta}}, \dot{\delta})$-basic terms.

Definition 2.7.10 (Number of Symbols of a $\mathrm{BPA}^{-}_{\mathrm{drt}}$ term)
We define $n(x)$, the number of symbols of x, inductively as follows:

(i). We define $n(\dot{\delta}) = 1$,

(ii). for $a \in A_{\delta}$, we define $n(\underline{\underline{a}}) = 1$,

(iii). for closed $\mathrm{BPA}_{\mathrm{drt}}$ terms x and y, we define $n(x + y) = n(x \cdot y) = n(x) + n(y) + 1$,

(iv). for a closed $\mathrm{BPA}_{\mathrm{drt}}$ term x, we define $n(\sigma_{\mathrm{rel}}(x)) = n(\nu_{\mathrm{rel}}(x)) = n(x) + 1$.

Theorem 2.7.11 (Elimination for $\mathrm{BPA}^{-}_{\mathrm{drt}}$)
Let t be a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$ term. Then there is a basic term s such that $\mathrm{BPA}^{-}_{\mathrm{drt}} \vdash t = s$.

$$
\begin{array}{rll}
(x+y)\cdot z \to x\cdot z + y\cdot z & & \text{RA4}\\
(x\cdot y)\cdot z \to x\cdot (y\cdot z) & & \text{RA5}\\
\sigma_{\mathrm{rel}}(x)\cdot y \to \sigma_{\mathrm{rel}}(x\cdot y) & & \text{RDRT2}\\
\dot{\delta}\cdot x \to \dot{\delta} & & \text{RA7ID}\\
\nu_{\mathrm{rel}}(\underline{\underline{a}}) \to \underline{\underline{a}} & & \text{RDCS1}\\
\nu_{\mathrm{rel}}(x+y) \to \nu_{\mathrm{rel}}(x) + \nu_{\mathrm{rel}}(y) & & \text{RDCS2}\\
\nu_{\mathrm{rel}}(x\cdot y) \to \nu_{\mathrm{rel}}(x)\cdot y & & \text{RDCS3}\\
\nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x)) \to \underline{\underline{\delta}} & & \text{RDCS4}\\
\nu_{\mathrm{rel}}(\dot{\delta}) \to \dot{\delta} & & \text{RDCSID}
\end{array}
$$

Table 21: Term rewriting system for $\mathrm{BPA}^{-}_{\mathrm{drt}}$.

Proof The term rewriting system of Table 21 is associated to $\mathrm{BPA}^{-}_{\mathrm{drt}}$ by assigning a direction to the axioms. With the method of the lexicographical path ordering it is easily proven that this term rewriting system is strongly normalizing. Give $\cdot$ the lexicographical status for the first argument and define the following well-founded partial ordering on constant and function symbols:

$$
\begin{array}{ccccc}
\nu_{\mathrm{rel}} & > & \cdot & > & +\\
\vee & & \vee & & \\
\underline{\underline{\delta}} & & \sigma_{\mathrm{rel}} & &
\end{array}
$$

We give the following reductions for the rewriting rules RA7ID and RDCSID:

$$
\begin{aligned}
\dot{\delta}\cdot x &\succ_{\mathrm{lpo}} \dot{\delta}\cdot^{\star} x\\
&\succ_{\mathrm{lpo}} \dot{\delta}\\
\nu_{\mathrm{rel}}(\dot{\delta}) &\succ_{\mathrm{lpo}} \nu_{\mathrm{rel}}{}^{\star}(\dot{\delta})\\
&\succ_{\mathrm{lpo}} \dot{\delta}
\end{aligned}
$$

Note that the reductions for the other rewriting rules have already been given in the proofs of previous elimination theorems.

Next, we will prove that the normal forms of the closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$ terms are basic terms. Thereto, suppose that s is a normal form of some closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$ term. Furthermore, suppose that s is not a basic term. Let s' denote the smallest subterm of s which is not a basic term. Then we can prove that s' is not a normal form by case analysis. We distinguish all possible cases:

(i). s' is an atomic action, $\underline{\underline{\delta}}$, or $\dot{\delta}$. But then s' is a basic term. This is in contradiction with the assumption that s' is not a basic term, so this case does not occur.

(ii). s' is of the form $s'_1 \cdot s'_2$ for basic terms s'_1 and s'_2. With case analysis on the structure of basic term s'_1:

(a) If s'_1 is $\dot{\delta}$ then rewriting rule RA7ID can be applied, and hence s' is not a normal form.

(b) If s'_1 is of the form $\underline{\underline{a}}$ for some $a \in A_\delta$, then $s'_1 \cdot s'_2$ is a basic term, and so s' is a basic term which again contradicts the assumption that s' is not a basic term. This case can therefore not occur.

(c) If s'_1 is of the form $\underline{\underline{a}} \cdot t$ for some $a \in A_\delta$ and some basic term t, then rewriting rule RA5 can be applied. So, s' is not a normal form.

(d) If s'_1 is of the form $t_1 + t_2$ for t_1 and t_2 basic terms. Then rewriting rule RA4 is applicable. Therefore, s' is not a normal form.

(e) If s'_1 is of the form $\sigma_{\mathrm{rel}}(t)$ for some basic term t. Then rewriting rule RDRT2 is applicable. So, s' is not a normal form.

(iii). s' is of the form $s'_1 + s'_2$ for basic terms s'_1 and s'_2. In this case s' would be a basic term, which contradicts the assumption that s' is not a basic term. Therefore, this case cannot occur.

(iv). s' is of the form $\sigma_{\mathrm{rel}}(t)$ for some basic term t. But then s' is basic term too, so the case does not occur.

(v). s' is of the form $\nu_{\mathrm{rel}}(t)$ for some basic term t. But then one of RDCS1–RDCS4 or RDCSID is applicable, so s' is not a normal form.

In any case that can occur it follows that s' is not a normal form. Since s' is a subterm of s, we conclude that s is not a normal form. This contradicts the assumption that s is a normal form. From this contradiction we conclude that s is a basic term, which completes the proof. ■

Remark 2.7.12 (Elimination for BPA$^-_{\mathrm{drt}}$)
Elimination for a slightly different version of BPA$^-_{\mathrm{drt}}$ is also claimed (without proof) in Section 3.4 of [10].

Theorem 2.7.13 (Soundness of BPA$^-_{\mathrm{drt}}$)
The set of closed BPA$^-_{drt}$ terms modulo bisimulation equivalence is a model of BPA$^-_{drt}$.

Proof For soundness of Axioms A1–A5, DRT1–DRT4, and DCS1–DCS4 we refer to the proof of soundness of BPA$^-_{\mathrm{drt}}$-ID. To extend these proofs from BPA$^-_{\mathrm{drt}}$-ID to BPA$^-_{\mathrm{drt}}$, we have to check that the bisimulations given in previous soundness proofs respect the ID predicate (as required by transfer condition (iv.) in 2.7.7 on page 46). However, as the fact that they do can be easily checked, we will not give details.

It remains to prove soundness of the axioms from Table 19 on page 46. For all axioms, we look at the transitions of both sides at the same time.

Axiom DRTSID Take the relation:

$$R = \{(\sigma_{\mathrm{rel}}(\dot{\delta}), \underline{\underline{\delta}})\}$$

We look at the transitions of both sides at the same time. We have $\sigma_{\mathrm{rel}}(\dot{\delta}) \not\rightarrow$ and $\underline{\underline{\delta}} \not\rightarrow$. Also, $\neg\mathrm{ID}(\sigma_{\mathrm{rel}}(\dot{\delta}))$ and $\neg\mathrm{ID}(\underline{\underline{\delta}})$.

Axiom A6ID Take the relation:

$$R = \{(s, s), (s + \dot{\delta}, s) \mid s \in C(\mathrm{BPA}^-_{\mathrm{drt}})\}$$

We look at the transitions of both sides at the same time. We have $s + \dot{\delta} \rightarrow p$ iff $s \rightarrow p$, and note that $(p, p) \in R$. Also, $\mathrm{ID}(s + \dot{\delta})$ iff $\mathrm{ID}(s) \wedge \mathrm{ID}(\dot{\delta})$ iff $\mathrm{ID}(s)$.

Axiom A7ID Take the relation:

$$R = \{(\dot{\delta} \cdot s, \dot{\delta}) \mid s \in C(\mathrm{BPA}^{-}_{\mathrm{drt}})\}$$

We look at the transitions of both sides at the same time. We have $\dot{\delta} \cdot s \not\rightarrow$ and $\dot{\delta} \not\rightarrow$. Also, $\mathrm{ID}(\dot{\delta} \cdot s)$ and $\mathrm{ID}(\dot{\delta})$.

Axiom DCSID Take the relation:

$$R = \{(\nu_{\mathrm{rel}}(\dot{\delta}), \dot{\delta})\}$$

We look at the transitions of both sides at the same time. We have $\nu_{\mathrm{rel}}(\dot{\delta}) \not\rightarrow$ and $\dot{\delta} \not\rightarrow$. Also, $\mathrm{ID}(\nu_{\mathrm{rel}}(\dot{\delta}))$ and $\mathrm{ID}(\dot{\delta})$.

■

Remark 2.7.14 (Soundness of $\mathrm{BPA}^{-}_{\mathrm{drt}}$)
Soundness of a slightly different version of $\mathrm{BPA}^{-}_{\mathrm{drt}}$ is also claimed (without proof) in Section 3.5 of [10].

Lemma 2.7.15 (Towards Completeness of $\mathrm{BPA}^{-}_{\mathrm{drt}}$)
Let x be a closed BPA^{-}_{drt} term and let $a \in A$. Then we have:

(i). $T(BPA^{-}_{drt}) \models x \xrightarrow{a} \surd \implies BPA^{-}_{drt} \vdash x = \underline{\underline{a}} + x,$

(ii). $T(BPA^{-}_{drt}) \models x \xrightarrow{a} y \implies BPA^{-}_{drt} \vdash x = \underline{\underline{a}} \cdot y + x,$

(iii). $T(BPA^{-}_{drt}) \models ID(x) \implies BPA^{-}_{drt} \vdash x = \dot{\delta},$

(iv). $T(BPA^{-}_{drt}) \models \neg ID(x) \implies BPA^{-}_{drt} \vdash x + \underline{\underline{\delta}} = x,$

(v). $T(BPA^{-}_{drt}) \models x \overset{\sigma}{\not\rightarrow} \implies BPA^{-}_{drt} \vdash x = \nu_{rel}(x),$

(vi). $T(BPA^{-}_{drt}) \models x \xrightarrow{\sigma} y \implies BPA^{-}_{drt} \vdash x = \sigma_{rel}(y) + \nu_{rel}(x),$

(vii). $T(BPA^{-}_{drt}) \models x \xrightarrow{a} y \implies n(x) > n(y),$

(viii). $T(BPA^{-}_{drt}) \models x \xrightarrow{\sigma} y \implies n(x) > n(y).$

Proof For part (i)–(vi) we assume, by Theorem 2.7.11 and Theorem 2.7.13, without loss of generality, that x is a basic term, and apply induction on the structure of basic terms. For part (vii) and (viii) we again have to use induction on the general structure of terms.

(i). Suppose that $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models x \xrightarrow{a} \surd$. Case 1: $x \equiv \dot{\delta}$. This is in contradiction with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models x \xrightarrow{a} \surd$, so this case does not occur. Case 2: $x \equiv \underline{\underline{b}}$, where $b \in A_{\delta}$. Because $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models x \xrightarrow{a} \surd$, it must be the case that $b \equiv a$. So we have $\mathrm{BPA}^{-}_{\mathrm{drt}} \vdash x = \underline{\underline{b}} = \underline{\underline{b}} + \underline{\underline{b}} = \underline{\underline{a}} + \underline{\underline{b}} = \underline{\underline{a}} + x$. Case 3: $x \equiv \underline{\underline{b}} \cdot x'$, where $b \in A_{\delta}$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models x \xrightarrow{a} \surd$, so this case does not occur. Case 4: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models x \xrightarrow{a} \surd$, necessarily $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models x' \xrightarrow{a} \surd$ or $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models x'' \xrightarrow{a} \surd$. Therefore, by the induction hypothesis, $\mathrm{BPA}^{-}_{\mathrm{drt}} \vdash x' = \underline{\underline{a}} + x'$ or $\mathrm{BPA}^{-}_{\mathrm{drt}} \vdash x'' = \underline{\underline{a}} + x''$. But then in both cases $\mathrm{BPA}^{-}_{\mathrm{drt}} \vdash x = x' + x'' = \underline{\underline{a}} + x' + x'' = \underline{\underline{a}} + x$. Case 5: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\mathrm{BPA}^{-}_{\mathrm{drt}}) \models x \xrightarrow{a} \surd$, so this case does not occur.

(ii). Suppose that $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{a} y$. Case 1: $x \equiv \dot{\delta}$. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{a} y$, so this case does not occur. Case 2: $x \equiv \underline{\underline{b}}$, where $b \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{a} y$, so this case does not occur. Case 3: $x \equiv \underline{\underline{b}} \cdot x'$, where $b \in A_\delta$ and x' is a basic term. Then, because $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{a} y$, it must be that $b \equiv a$ and $x' \equiv y$. So, $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x = x + x = \underline{\underline{b}} \cdot x' + x = \underline{\underline{a}} \cdot y + x$. Case 4: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{a} y$, necessarily $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x' \xrightarrow{a} y$ or $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x'' \xrightarrow{a} y$. Therefore, by the induction hypothesis, $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x' = \underline{\underline{a}} \cdot y + x'$ or $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x'' = \underline{\underline{a}} \cdot y + x''$. But then in both cases $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x = x' + x'' = \underline{\underline{a}} \cdot y + x' + x'' = \underline{\underline{a}} \cdot y + x$. Case 5: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{a} y$, so this case does not occur.

(iii). Suppose that $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \mathrm{ID}(x)$. Case 1: $x \equiv \dot{\delta}$. Then we have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x = \dot{\delta}$ is trivially fulfilled. Case 2: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \mathrm{ID}(x)$, so this case does not occur. Case 3: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \mathrm{ID}(x)$, so this case does not occur. Case 4: $x \equiv x' + x''$, where x' and x'' are basic terms. Then, because $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \mathrm{ID}(x)$, it must be the case that $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \mathrm{ID}(x'), \mathrm{ID}(x'')$. So, by the induction hypothesis, we have that $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x' = \dot{\delta}, x'' = \dot{\delta}$. But then also $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x = x' + x'' = \dot{\delta} + \dot{\delta} = \dot{\delta}$. Case 5: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \mathrm{ID}(x)$, so this case does not occur.

(iv). Suppose that $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \neg\mathrm{ID}(x)$. Case 1: $x \equiv \dot{\delta}$. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \neg\mathrm{ID}(x)$, so this case does not occur. Case 2: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. Then we have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x + \underline{\underline{\delta}} = \underline{\underline{a}} + \underline{\underline{\delta}} = \underline{\underline{a}} = x$. Case 3: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. Then $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x + \underline{\underline{\delta}} = \underline{\underline{a}} \cdot x' + \underline{\underline{\delta}} = \underline{\underline{a}} \cdot x' + \underline{\underline{\delta}} \cdot x' = (\underline{\underline{a}} + \underline{\underline{\delta}}) \cdot x' = \underline{\underline{a}} \cdot x' = x$. Case 4: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \neg\mathrm{ID}(x)$, necessarily $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \neg\mathrm{ID}(x')$ or $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \neg\mathrm{ID}(x'')$. Therefore, by the induction hypothesis, $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x' + \underline{\underline{\delta}} = x'$ or $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x'' + \underline{\underline{\delta}} = x''$. So, in both cases, $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x + \underline{\underline{\delta}} = x' + x'' + \underline{\underline{\delta}} = x' + x'' = x$. Case 5: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. Then $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(x') + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(x') = x$.

(v). Suppose that $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \overset{\sigma}{\nrightarrow}$. Case 1: $x \equiv \dot{\delta}$. By Axiom DCSID we have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x = \dot{\delta} = \nu_{\mathrm{rel}}(\dot{\delta}) = \nu_{\mathrm{rel}}(x)$. Case 2: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. We have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x = \underline{\underline{a}} = \nu_{\mathrm{rel}}(\underline{\underline{a}}) = \nu_{\mathrm{rel}}(x)$. Case 3: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. We have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x = \underline{\underline{a}} \cdot x' = \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot x' = \nu_{\mathrm{rel}}(\underline{\underline{a}} \cdot x') = \nu_{\mathrm{rel}}(x)$. Case 4: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \overset{\sigma}{\nrightarrow}$, necessarily $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x' \overset{\sigma}{\nrightarrow}$ and $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x'' \overset{\sigma}{\nrightarrow}$. Therefore, by the induction hypothesis, $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x' = \nu_{\mathrm{rel}}(x')$ and $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x'' = \nu_{\mathrm{rel}}(x'')$. But then also $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x = x' + x'' = \nu_{\mathrm{rel}}(x') + \nu_{\mathrm{rel}}(x'') = \nu_{\mathrm{rel}}(x' + x'') = \nu_{\mathrm{rel}}(x)$. Case 5: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term with $\neg\mathrm{ID}(x)$. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \overset{\sigma}{\nrightarrow}$, so this case does not occur. Case 6: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. Then, by $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \overset{\sigma}{\nrightarrow}$, it must be the case that $\mathrm{ID}(x')$. So, by (iii), we have that $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x' = \dot{\delta}$. Therefore, $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x = \sigma_{\mathrm{rel}}(x') = \sigma_{\mathrm{rel}}(\dot{\delta}) = \underline{\underline{\delta}} = \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x')) = \nu_{\mathrm{rel}}(x)$.

(vi). Suppose that $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$. Case 1: $x \equiv \dot{\delta}$. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 2: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 3:

$x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 4: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, necessarily (1) $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x' \xrightarrow{\sigma} y, x'' \stackrel{\sigma}{\nrightarrow}$, or, (2) $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x' \stackrel{\sigma}{\nrightarrow}, x'' \xrightarrow{\sigma} y$, or, (3) $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x' \xrightarrow{\sigma} y', x'' \xrightarrow{\sigma} y''$ where $y \equiv y' + y''$. In the first case, by the induction hypothesis, we have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x')$, and, by (v), $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x'' = \nu_{\mathrm{rel}}(x'')$. Therefore, $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x = x' + x'' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x') + \nu_{\mathrm{rel}}(x'') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x' + x'') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$. The second case is treated analogously. In the third case we have, by the induction hypothesis, $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x' = \sigma_{\mathrm{rel}}(y') + \nu_{\mathrm{rel}}(x'), x'' = \sigma_{\mathrm{rel}}(y'') + \nu_{\mathrm{rel}}(x'')$. Therefore we have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x = x' + x'' = \sigma_{\mathrm{rel}}(y') + \nu_{\mathrm{rel}}(x') + \sigma_{\mathrm{rel}}(y'') + \nu_{\mathrm{rel}}(x'') = \sigma_{\mathrm{rel}}(y' + y'') + \nu_{\mathrm{rel}}(x' + x'') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$. Case 5: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. Because $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, it must be the case that $x' \equiv y$. So we have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash x = \sigma_{\mathrm{rel}}(x') = \sigma_{\mathrm{rel}}(y) = \sigma_{\mathrm{rel}}(y) + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x')) = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$.

(vii). Suppose that $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{a} y$. Case 1: $x \equiv \dot{\delta}$. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 2: $x \equiv \underline{\underline{b}}$, where $b \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{a} y$, so this case does not occur. Case 3: $x \equiv x' \cdot x''$, for certain terms x' and x''. Then, because $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{a} y$, we either have $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x' \xrightarrow{a} \surd$ and $y \equiv x''$, or we have $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x' \xrightarrow{a} x'''$ and $y \equiv x''' \cdot x''$ for some term x'''. In the first case, we have $n(x) = n(x' \cdot x'') = n(x') + n(x'') + 1 > n(x'') = n(y)$, and in the second we can apply the induction hypothesis to arrive at $n(x') > n(x''')$, so we get $n(x) = n(x' \cdot x'') = n(x') + n(x'') + 1 > n(x''') + n(x') + 1 = n(x''' \cdot x'') = n(y)$. Case 4: $x \equiv x' + x''$, for certain terms x' and x''. Since $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{a} y$, necessarily $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x' \xrightarrow{a} y$ or $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x'' \xrightarrow{a} y$. Therefore, by the induction hypothesis, $n(x') > n(y)$ or $n(x'') > n(y)$. In both cases $n(x) = n(x' + x'') = n(x') + n(x'') + 1 > n(y)$. Case 5: $x \equiv \sigma_{\mathrm{rel}}(x')$, for a certain term x'. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{a} y$, so this case does not occur. Case 6: $x \equiv \nu_{\mathrm{rel}}(x')$, for a certain term x'. Since $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{a} y$, necessarily $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x' \xrightarrow{a} y$. Therefore, by the induction hypothesis, $n(x') > n(y)$. So, $n(x) = n(\nu_{\mathrm{rel}}(x')) = n(x') + 1 > n(y)$.

(viii). Suppose that $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$. Case 1: $x \equiv \dot{\delta}$. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 2: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 3: $x \equiv x' \cdot x''$, for certain terms x' and x''. Then necessarily, $x' \xrightarrow{\sigma} x'''$ and $y \equiv x''' \cdot x''$ for some term x'''. We now can apply the induction hypothesis to arrive at $n(x') > n(x''')$, so we get $n(x) = n(x' \cdot x'') = n(x') + n(x'') + 1 > n(x''') + n(x'') + 1 = n(x''' \cdot x'') = n(y)$. Case 4: $x \equiv x' + x''$, for certain terms x' and x''. As $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, necessarily (1) $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x' \xrightarrow{\sigma} y, x'' \stackrel{\sigma}{\nrightarrow}$, or, (2) $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x' \stackrel{\sigma}{\nrightarrow}, x'' \xrightarrow{\sigma} y$, or, (3) $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x' \xrightarrow{\sigma} y', x'' \xrightarrow{\sigma} y''$ where $y \equiv y' + y''$. In the first case, by the induction hypothesis, $n(x') > n(y)$. So $n(x) = n(x' + x'') = n(x') + n(x'') + 1 > n(y)$. The second case is treated analogously. In the third case, by the induction hypothesis, $n(x') > n(y')$ and $n(x'') > n(y'')$. So $n(x) = n(x' + x'') = n(x') + n(x'') + 1 > n(y') + n(y'') + 1 = n(y)$. Case 5: $x \equiv \sigma_{\mathrm{rel}}(x')$, for a certain term x'. Because $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, it must be the case that $x' \equiv y$. Then we have $n(x) = n(\sigma_{\mathrm{rel}}(x')) = n(x') + 1 = n(y) + 1 > n(y)$. Case 6: $x \equiv \nu_{\mathrm{rel}}(x')$, for a certain term x'. This is in contradiction with $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur.

■

Theorem 2.7.16 (Completeness of $\mathrm{BPA}^-_{\mathrm{drt}}$)
The axiom system BPA^-_{drt} is a complete axiomatization of the set of closed BPA^-_{drt} terms modulo (strong) bisimulation equivalence.

Proof Suppose $s + t \sim_{\mathrm{BPA}^-_{\mathrm{drt}}} t$. We will prove, with induction on the structure of basic term s, that $\mathrm{BPA}^-_{\mathrm{drt}} \vdash s + t = t$. By Theorem 2.7.11 we can restrict ourselves to basic terms without loss of generality. The proof is done with induction on $n(s)$, using Lemma 2.7.15(vii)–(viii) and case distinction on the form of basic term s.

(i). $s \equiv \dot{\delta}$. Using Axiom A6ID we have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash s + t = \dot{\delta} + t = t + \dot{\delta} = t$.

(ii). $s \equiv \underline{\underline{\delta}}$. Then, since for $u \in A_\sigma$ we have $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models s \stackrel{u}{\nrightarrow}$, also $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models s + t \xrightarrow{u} p$ iff $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models t \xrightarrow{u} p$, and for $a \in A$, $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models s + t \xrightarrow{a} \surd$ iff $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models t \xrightarrow{a} \surd$. Furthermore, we have $\neg\mathrm{ID}(s + t)$, since $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \neg\mathrm{ID}(s)$. Since $s + t \sim_{\mathrm{BPA}^-_{\mathrm{drt}}} t$, we also have $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \neg\mathrm{ID}(t)$. Using Lemma 2.7.15(iv) we have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash s + t = \underline{\underline{\delta}} + t = t + \underline{\underline{\delta}} = t$.

(iii). $s \equiv \underline{\underline{a}}$, where $a \in A$. From the deduction rules we have $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models s \xrightarrow{a} \surd$ and $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models s+t \xrightarrow{a} \surd$. Since $s+t \sim_{\mathrm{BPA}^-_{\mathrm{drt}}} t$ we also have $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models t \xrightarrow{a} \surd$. By Lemma 2.7.15(i) we obtain $\mathrm{BPA}^-_{\mathrm{drt}} \vdash t = \underline{\underline{a}} + t$. So, $\mathrm{BPA}^-_{\mathrm{drt}} \vdash s + t = \underline{\underline{a}} + t = t$.

(iv). $s \equiv \underline{\underline{\delta}} \cdot s'$, where s' is a basic term. Then we have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash s = \underline{\underline{\delta}} \cdot s' = \underline{\underline{\delta}}$ and, using (ii), $\mathrm{BPA}^-_{\mathrm{drt}} \vdash s + t = t$.

(v). $s \equiv \underline{\underline{a}} \cdot s'$, where $a \in A$ and s' is a basic term. From the deduction rules we obtain $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models s \xrightarrow{a} s'$ and $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models s + t \xrightarrow{a} s'$. Since $s + t \sim_{\mathrm{BPA}^-_{\mathrm{drt}}} t$, we then also have $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models t \xrightarrow{a} t'$ for some t' such that $s' \sim_{\mathrm{BPA}^-_{\mathrm{drt}}} t'$. By the induction hypothesis we have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash s' = t'$. From Lemma 2.7.15(ii) we have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash t = \underline{\underline{a}} \cdot t' + t$. So, $\mathrm{BPA}^-_{\mathrm{drt}} \vdash s + t = \underline{\underline{a}} \cdot s' + t = \underline{\underline{a}} \cdot t' + t = t$.

(vi). $s \equiv s' + s''$, where s' and s'' are basic terms. Since $s' + s'' + t \sim_{\mathrm{BPA}^-_{\mathrm{drt}}} t$, we also have $s' + t \sim_{\mathrm{BPA}^-_{\mathrm{drt}}} t$ and $s'' + t \sim_{\mathrm{BPA}^-_{\mathrm{drt}}} t$. By the induction hypothesis we then have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash s' + t = t, s'' + t = t$. So, $\mathrm{BPA}^-_{\mathrm{drt}} \vdash s + t = s' + s'' + t = s' + t = t$.

(vii). $s \equiv \sigma_{\mathrm{rel}}(s')$, where s' is a basic term. From the deduction rules we have $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models \sigma_{\mathrm{rel}}(s') \xrightarrow{\sigma} s'$ and since $s + t \sim_{\mathrm{BPA}^-_{\mathrm{drt}}} t$ we also have $T(\mathrm{BPA}^-_{\mathrm{drt}}) \models t \xrightarrow{\sigma} t', s + t \xrightarrow{\sigma} s' + t'$ for some t' such that $s' + t' \sim_{\mathrm{BPA}^-_{\mathrm{drt}}} t'$. By Lemma 2.7.15(vi) we have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash t = \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t)$. By the induction hypothesis we have $\mathrm{BPA}^-_{\mathrm{drt}} \vdash s' + t' = t'$. So, $\mathrm{BPA}^-_{\mathrm{drt}} \vdash s + t = \sigma_{\mathrm{rel}}(s') + t = \sigma_{\mathrm{rel}}(s') + \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(s' + t') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t) = t$.

■

Remark 2.7.17 (Completeness of $\mathrm{BPA}^-_{\mathrm{drt}}$)
Completeness of a slightly different version of $\mathrm{BPA}^-_{\mathrm{drt}}$ is also claimed (without proof) in Section 3.5 of [10].

2.8 Soundness and Completeness of $\mathrm{BPA}^{+}_{\mathrm{drt}}$

Definition 2.8.1 (Signature of $\mathrm{BPA}_{\mathrm{drt}}$)
The signature of $\mathrm{BPA}_{\mathrm{drt}}$ consists of the *undelayable atomic actions* $\{\underline{\underline{a}}|a \in A\}$, the *delayable deadlock actions* $\{a|a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *delayable deadlock constant* δ, the *immediate deadlock constant* $\dot{\delta}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *time unit delay operator* σ_{rel}, the *"now" operator* ν_{rel}, and the *unbounded start delay operator* $\lfloor\ \rfloor^{\omega}$.

Definition 2.8.2 (Axioms of $\mathrm{BPA}_{\mathrm{drt}}$)
The process algebra $\mathrm{BPA}_{\mathrm{drt}}$ is axiomatized by the axioms of $\mathrm{BPA}^{-}_{\mathrm{drt}}$ given in Definition 2.7.2 on page 45, and Axioms ATS and USD shown in Table 14 on page 32: $\mathrm{BPA}_{\mathrm{drt}}$ = A1–A5 + A6ID + A7ID + DRT1–DRT5 + DRTSID + DCS1–DCS4 + DCSID + ATS + USD.

Definition 2.8.3 (Recursion Principle for $\mathrm{BPA}_{\mathrm{drt}}$)
Next to the axioms mentioned in Definition 2.8.2, the system $\mathrm{BPA}^{+}_{\mathrm{drt}}$ also contains the *recursion principle* RSP(USD) shown in Table 15 on page 33. For more information on recursion principles and their status with respect to axioms, see [14].

Definition 2.8.4 (Semantics of $\mathrm{BPA}_{\mathrm{drt}}$)
The semantics of $\mathrm{BPA}_{\mathrm{drt}}$ are given by the term deduction system $T(\mathrm{BPA}_{\mathrm{drt}})$ induced by the deduction rules for $\mathrm{BPA}^{-}_{\mathrm{drt}}$ given in Definition 2.7.6 on page 46 and the deduction rules given in Table 16 on page 33.

Remark 2.8.5 (Semantics of $\mathrm{BPA}_{\mathrm{drt}}$)
Note that for any closed $\mathrm{BPA}_{\mathrm{drt}}$ term x, we have that $\neg\mathrm{ID}(\lfloor x\rfloor^{\omega})$. Hence, there is no deduction rule for $\mathrm{ID}(\lfloor x\rfloor^{\omega})$.

Definition 2.8.6 (Bisimulation and Bisimulation Model for $\mathrm{BPA}_{\mathrm{drt}}$)
Bisimulation for $\mathrm{BPA}_{\mathrm{drt}}$ and the corresponding bisimulation model are defined in the same way as for $\mathrm{BPA}^{-}_{\mathrm{drt}}$ and BPA respectively. Replace "$\mathrm{BPA}^{-}_{\mathrm{drt}}$" by "$\mathrm{BPA}_{\mathrm{drt}}$" in Definition 2.7.7 on page 46 and "BPA" by "$\mathrm{BPA}_{\mathrm{drt}}$" in Definition 2.2.11 on page 8.

Definition 2.8.7 (Basic Terms of $\mathrm{BPA}_{\mathrm{drt}}$)
We define $(\sigma, \underline{\underline{\delta}}, \delta, \dot{\delta})$*-basic terms* inductively as follows:

(i). Immediate deadlock $\dot{\delta}$ is a $(\sigma, \underline{\underline{\delta}}, \delta, \dot{\delta})$-basic term,

(ii). If $a \in A_{\delta}$, then $\underline{\underline{a}}$ and a are $(\sigma, \underline{\underline{\delta}}, \delta, \dot{\delta})$-basic terms,

(iii). if $a \in A_{\delta}$ and t is a $(\sigma, \underline{\underline{\delta}}, \delta, \dot{\delta})$-basic term, then $\underline{\underline{a}} \cdot t$ and $a \cdot t$ are $(\sigma, \underline{\underline{\delta}}, \delta, \dot{\delta})$-basic terms,

(iv). if t and s are $(\sigma, \underline{\underline{\delta}}, \delta, \dot{\delta})$-basic terms, then $t + s$ is a $(\sigma, \underline{\underline{\delta}}, \delta, \dot{\delta})$-basic term,

(v). if t is a $(\sigma, \underline{\underline{\delta}}, \delta, \dot{\delta})$-basic term, then $\sigma_{\mathrm{rel}}(t)$ is a $(\sigma, \underline{\underline{\delta}}, \delta, \dot{\delta})$-basic term.

From now on, if we speak of basic terms in the context of $\mathrm{BPA}_{\mathrm{drt}}$, we mean $(\sigma, \underline{\underline{\delta}}, \delta, \dot{\delta})$-basic terms.

Theorem 2.8.8 (General Form of Basic Terms of BPA$_{\text{drt}}$)
Modulo the commutativity and associativity of the $+$, and modulo superfluous $\dot{\delta}$ summands, all basic terms t of BPA$_{drt}$ are of the form:

$$t \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l + \sum_{o<r} \sigma_{rel}(v_o)$$

for $m, n, p, q, r \in \mathbb{N}$, $a_i, b_j, c_k, d_l \in A_\delta$, and basic terms s_i, u_l, and v_o.

Proof Trivial, by inspection of the definition of basic terms, Definition 2.8.7. Observe that the general form of basic terms is closed under the formation rules gives in Definition 2.8.7. ∎

Remark 2.8.9 (General Form of Basic Terms of BPA$_{\text{drt}}$)
Note that the case $t = \dot{\delta}$ is generated when we take $m = n = p = q = r = 0$. See also Definition 2.7.5.

Definition 2.8.10 (Number of Symbols of a BPA$_{\text{drt}}$ term)
We define $n(x)$, the number of symbols of x, inductively as follows:

(i). We define $n(\dot{\delta}) = 1$,

(ii). for $a \in A_\delta$, we define $n(\underline{\underline{a}}) = n(a) = 1$,

(iii). for closed BPA$_{\text{drt}}$ terms x and y, we define $n(x + y) = n(x \cdot y) = n(x) + n(y) + 1$,

(iv). for a closed BPA$_{\text{drt}}$ term x, we define $n(\sigma_{\text{rel}}(x)) = n(\nu_{\text{rel}}(x)) = n(\lfloor x \rfloor^\omega) = n(x) + 1$.

Proposition 2.8.11 (Properties of BPA$^+_{\text{drt}}$, Part I)
For BPA$_{drt}$ terms x and y, and any $a \in A_\delta$, we have the following equalities:

(i). $BPA^+_{drt} \vdash \lfloor a \rfloor^\omega = a$

(ii). $BPA^+_{drt} \vdash \lfloor x \cdot y \rfloor^\omega = \lfloor x \rfloor^\omega \cdot y$

(iii). $BPA^+_{drt} \vdash \lfloor x + y \rfloor^\omega = \lfloor x \rfloor^\omega + \lfloor y \rfloor^\omega$

(iv). $BPA^+_{drt} \vdash \lfloor \sigma_{rel}(x) \rfloor^\omega = \delta$

(v). $BPA^+_{drt} \vdash \lfloor \dot{\delta} \rfloor^\omega = \delta$

(vi). $BPA_{drt} \vdash \nu_{rel}(a) = \underline{\underline{a}}$

(vii). $BPA_{drt} \vdash \lfloor x \rfloor^\omega + \underline{\underline{\delta}} = \lfloor x \rfloor^\omega$

Proof The proofs for equality (i)–(iv) and (vi) given in Proposition 2.6.9 on page 34, with respect to BPA$_{\text{drt}}$–ID, remain valid in the setting of BPA$_{\text{drt}}$, as can be easily checked.

Equality (v) and (vii) do not appear in Proposition 2.6.9. Consider the following computation for equality (v):

$$\begin{aligned} \text{BPA}_{\text{drt}} \vdash \delta &= \lfloor \underline{\underline{\delta}} \rfloor^\omega \\ &= \nu_{\text{rel}}(\underline{\underline{\delta}}) + \sigma_{\text{rel}}(\lfloor \underline{\underline{\delta}} \rfloor^\omega) \\ &= \underline{\underline{\delta}} + \sigma_{\text{rel}}(\delta) \\ &= \sigma_{\text{rel}}(\delta) \\ &= \dot{\delta} + \sigma_{\text{rel}}(\delta) \\ &= \nu_{\text{rel}}(\dot{\delta}) + \sigma_{\text{rel}}(\delta) \end{aligned}$$

Using RSP(USD) we obtain:

$$\mathrm{BPA}^+_{\mathrm{drt}} \vdash \delta = \lfloor \dot{\delta} \rfloor^{\omega}$$

Finally, consider the following computation for equality (vii):

$$\mathrm{BPA}_{\mathrm{drt}} \vdash \lfloor x \rfloor^{\omega} + \underline{\underline{\delta}} = \nu_{\mathrm{rel}}(x) + \sigma_{\mathrm{rel}}(\lfloor x \rfloor^{\omega}) + \underline{\underline{\delta}} = \nu_{\mathrm{rel}}(x) + \sigma_{\mathrm{rel}}(\lfloor x \rfloor^{\omega}) = \lfloor x \rfloor^{\omega}$$

Note the use of Axiom DRT5 in the second step. ■

Proposition 2.8.12 (Properties of BPA$^+_{\mathrm{drt}}$, Part II)
For any BPA$_{drt}$ term x we have the following equality:

$$BPA^+_{drt} \vdash \delta \cdot x = \delta$$

Proof Using Proposition 2.8.11(ii) we derive:

$$\mathrm{BPA}^+_{\mathrm{drt}} \vdash \delta \cdot x = \lfloor \underline{\underline{\delta}} \rfloor^{\omega} \cdot x = \lfloor \underline{\underline{\delta}} \cdot x \rfloor^{\omega} = \lfloor \underline{\underline{\delta}} \rfloor^{\omega} = \delta$$

■

Lemma 2.8.13 (Representation of BPA$^+_{\mathrm{drt}}$ Terms)
Let t be a basic term. Then either:

(i). $BPA^+_{drt} \vdash t = \dot{\delta}$, or,

(ii). $BPA^+_{drt} \vdash t = \nu_{rel}(t) + \underline{\underline{\delta}}$, or,

(iii). $BPA^+_{drt} \vdash t = \lfloor t \rfloor^{\omega}$, or,

(iv). there exists a basic term s such that $BPA^+_{drt} \vdash t = \nu_{rel}(t) + \sigma_{rel}(s)$ and $n(s) < n(t)$.

Proof Let t be a basic term. By Theorem 2.5.9, we may now proceed by case analysis on the form of basic terms. Suppose, by Theorem 2.8.8, that t has the following general form:

$$t \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l + \sum_{o<r} \sigma_{\mathrm{rel}}(\nu_o)$$

for $m, n, p, q, r \in \mathbb{N}$, $a_i, b_j, c_k, d_l \in A_\delta$, and basic terms s_i, u_l, and ν_o. We distinguish four cases:

(i). There are no summands: $p = q = m = n = r = 0$.

(ii). Every, and at least one, summand starts with an undelayable action: $m + n \geq 1$ and $p = q = r = 0$.

(iii). Every, and at least one, summand starts with a delayable action: $p + q \geq 1$ and $m = n = r = 0$.

(iv). Neither of the above; there are both summands that start with delayable action and ones the start with undelayable actions, or there are summands that start with the time unit delay operator: $p + q + r \geq 1$ and $m + n + r \geq 1$

As can be easily seen, this covers all cases. We now prove that the four cases we distinguish exactly correspond to the four cases in the formulation of the theorem:

(i). We have $p = q = m = n = r = 0$. So, by Definition 2.7.5, $t \equiv \dot{\delta}$ and $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash t = \dot{\delta}$.

(ii). We have $m + n \geq 1$ and $p = q = r = 0$. So:

$$t \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}}$$

for $m, n \in \mathbb{N}$, $a_i, b_j \in A_\delta$, and basic terms s_i. Then we have the following computation:

$$\begin{aligned}
\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash t &= \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} = \sum_{i<m} \nu_{\mathrm{rel}}(\underline{\underline{a_i}}) \cdot s_i + \sum_{j<n} \nu_{\mathrm{rel}}(\underline{\underline{b_j}}) \\
&= \sum_{i<m} \nu_{\mathrm{rel}}(\underline{\underline{a_i}} \cdot s_i) + \sum_{j<n} \nu_{\mathrm{rel}}(\underline{\underline{b_j}}) = \nu_{\mathrm{rel}} \left(\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} \right) \\
&= \nu_{\mathrm{rel}} \left(\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} (\underline{\underline{b_j}} + \underline{\underline{\delta}}) \right) = \nu_{\mathrm{rel}} \left(\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \underline{\underline{\delta}} \right) \\
&= \nu_{\mathrm{rel}}(t + \underline{\underline{\delta}}) = \nu_{\mathrm{rel}}(t) + \nu_{\mathrm{rel}}(\underline{\underline{\delta}}) = \nu_{\mathrm{rel}}(t) + \underline{\underline{\delta}}
\end{aligned}$$

(iii). We have $p + q \geq 1$ and $m = n = r = 0$. So:

$$t \equiv \sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l$$

for $p, q \in \mathbb{N}$, $c_k, d_l \in A_\delta$, and basic terms u_k. Using Proposition 2.8.11(i)–(iii) we then have the following computation:

$$\begin{aligned}
\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash t &= \sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l = \sum_{k<p} \lfloor c_k \rfloor^{\omega} \cdot u_k + \sum_{l<q} \lfloor d_l \rfloor^{\omega} \\
&= \sum_{k<p} \lfloor c_k \cdot u_k \rfloor^{\omega} + \sum_{l<q} \lfloor d_l \rfloor^{\omega} = \left\lfloor \sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l \right\rfloor^{\omega} \\
&= \lfloor t \rfloor^{\omega}
\end{aligned}$$

(iv). We have $p + q + r \geq 1$ and $m + n + r \geq 1$. So:

$$t \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l + \sum_{o<r} \sigma_{\mathrm{rel}}(\nu_o)$$

for $m, n, p, q, r \in \mathbb{N}$, $a_i, b_j, c_k, d_l \in A_\delta$, and basic terms s_i, u_l, and ν_o. Then we have the following computation:

$$\begin{aligned}
\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash t &= \sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l + \sum_{o<r} \sigma_{\mathrm{rel}}(\nu_o) \\
&= \sum_{i<m} \nu_{\mathrm{rel}}(\underline{\underline{a_i}}) \cdot s_i + \sum_{j<n} \nu_{\mathrm{rel}}(\underline{\underline{b_j}}) + \\
&\quad \sum_{k<p} \lfloor \underline{\underline{c_k}} \rfloor^{\omega} \cdot u_k + \sum_{l<q} \lfloor \underline{\underline{d_l}} \rfloor^{\omega} + \sum_{o<r} \sigma_{\mathrm{rel}}(\nu_o)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i<m} \nu_{\text{rel}}(\underline{\underline{a_i}} \cdot s_i) + \sum_{j<n} \nu_{\text{rel}}(\underline{\underline{b_j}}) + \\
&\quad \sum_{k<p} (\nu_{\text{rel}}(\underline{\underline{c_k}}) + \sigma_{\text{rel}}(\lfloor \underline{\underline{c_k}} \rfloor^{\omega})) \cdot u_k + \sum_{l<q} (\nu_{\text{rel}}(\underline{\underline{d_l}}) + \sigma_{\text{rel}}(\lfloor \underline{\underline{d_l}} \rfloor^{\omega})) + \\
&\quad \sum_{o<r} \sigma_{\text{rel}}(\nu_o) \\
&= \sum_{i<m} \nu_{\text{rel}}(\underline{\underline{a_i}} \cdot s_i) + \sum_{j<n} \nu_{\text{rel}}(\underline{\underline{b_j}}) + \\
&\quad \sum_{k<p} (\nu_{\text{rel}}(\underline{\underline{c_k}}) + \sigma_{\text{rel}}(c_k)) \cdot u_k + \sum_{l<q} (\nu_{\text{rel}}(\underline{\underline{d_l}}) + \sigma_{\text{rel}}(d_l)) + \\
&\quad \sum_{o<r} \sigma_{\text{rel}}(\nu_o) \\
&= \sum_{i<m} \nu_{\text{rel}}(\underline{\underline{a_i}} \cdot s_i) + \sum_{j<n} \nu_{\text{rel}}(\underline{\underline{b_j}}) + \\
&\quad \sum_{k<p} (\nu_{\text{rel}}(\underline{\underline{c_k}}) \cdot u_k + \sigma_{\text{rel}}(c_k) \cdot u_k) + \sum_{l<q} (\nu_{\text{rel}}(\underline{\underline{d_l}}) + \sigma_{\text{rel}}(d_l)) + \\
&\quad \sum_{o<r} \sigma_{\text{rel}}(\nu_o) \\
&= \sum_{i<m} \nu_{\text{rel}}(\underline{\underline{a_i}} \cdot s_i) + \sum_{j<n} \nu_{\text{rel}}(\underline{\underline{b_j}}) + \\
&\quad \sum_{k<p} (\nu_{\text{rel}}(\underline{\underline{c_k}} \cdot u_k) + \sigma_{\text{rel}}(c_k \cdot u_k)) + \sum_{l<q} (\nu_{\text{rel}}(\underline{\underline{d_l}}) + \sigma_{\text{rel}}(d_l)) + \\
&\quad \sum_{o<r} \sigma_{\text{rel}}(\nu_o) \\
&= \nu_{\text{rel}}\left(\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \underline{\underline{c_k}} \cdot u_k + \sum_{l<q} \underline{\underline{d_l}} \right) + \\
&\quad \sigma_{\text{rel}}\left(\sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l + \sum_{o<r} \nu_o \right) \\
&= \nu_{\text{rel}}\Bigg(\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \\
&\qquad \sum_{k<p} (\underline{\underline{c_k}} \cdot u_k + \sigma_{\text{rel}}(\lfloor \underline{\underline{c_k}} \rfloor^{\omega}) \cdot u_k) + \sum_{l<q} (\underline{\underline{d_l}} + \sigma_{\text{rel}}(\lfloor \underline{\underline{d_l}} \rfloor^{\omega})) + \\
&\qquad \sum_{o<r} \sigma_{\text{rel}}(\nu_o) \Bigg) + \\
&\quad \sigma_{\text{rel}}\left(\sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l + \sum_{o<r} \nu_o \right) \\
&= \nu_{\text{rel}}\Bigg(\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \\
&\qquad \sum_{k<p} (\underline{\underline{c_k}} + \sigma_{\text{rel}}(\lfloor \underline{\underline{c_k}} \rfloor^{\omega})) \cdot u_k + \sum_{l<q} (\underline{\underline{d_l}} + \sigma_{\text{rel}}(\lfloor \underline{\underline{d_l}} \rfloor^{\omega})) +
\end{aligned}
$$

$$
\begin{aligned}
& \qquad \sum_{o<r} \sigma_{\mathrm{rel}}(\nu_o)\Bigg) + \\
& \sigma_{\mathrm{rel}}\left(\sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l + \sum_{o<r} \nu_o\right) \\
= {} & \nu_{\mathrm{rel}}\Bigg(\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \\
& \qquad \sum_{k<p} \lfloor \underline{\underline{c_k}} \rfloor^{\omega} \cdot u_k + \sum_{l<q} \lfloor \underline{\underline{d_l}} \rfloor^{\omega} + \sum_{o<r} \sigma_{\mathrm{rel}}(\nu_o)\Bigg) + \\
& \sigma_{\mathrm{rel}}\left(\sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l + \sum_{o<r} \nu_o\right) \\
= {} & \nu_{\mathrm{rel}}\left(\sum_{i<m} \underline{\underline{a_i}} \cdot s_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l + \sum_{o<r} \sigma_{\mathrm{rel}}(\nu_o)\right) + \\
& \sigma_{\mathrm{rel}}\left(\sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l + \sum_{o<r} \nu_o\right) \\
= {} & \nu_{\mathrm{rel}}(t) + \sigma_{\mathrm{rel}}(s)
\end{aligned}
$$

Where we define:

$$s \equiv \sum_{k<p} c_k \cdot u_k + \sum_{l<q} d_l + \sum_{o<r} \nu_o$$

Note that $n(s) < n(t)$ is now trivially satisfied: every summand of s also appears as a subterm of t, and by $m + n + r \geq 1$, t must contain summands that do not appear in s. Therefore, t must contain at least 2 more symbols than s.

■

Lemma 2.8.14 (Simplified Representation of $\mathbf{BPA}^{+}_{\mathbf{drt}}$ Terms)
Let t be a basic term. Then either:

(i). $BPA^{+}_{drt} \vdash t = \dot{\delta}$, *or,*

(ii). $BPA^{+}_{drt} \vdash t = t + \underline{\underline{\delta}}$.

Proof This lemma follows almost immediately from Lemma 2.8.13; case (i) mentioned there corresponds to case (i) here, and cases (ii)–(iv) mentioned there correspond to case (ii) here. We distinguish the four cases from Lemma 2.8.13:

(i). $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash t = \dot{\delta}$.

(ii). $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash t = \nu_{\mathrm{rel}}(t) + \underline{\underline{\delta}}$. Then we have, using Axiom A3:

$$\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash t = \nu_{\mathrm{rel}}(t) + \underline{\underline{\delta}} = \nu_{\mathrm{rel}}(t) + \underline{\underline{\delta}} + \underline{\underline{\delta}} = t + \underline{\underline{\delta}}.$$

(iii). $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash t = \lfloor t \rfloor^{\omega}$. Then we have, using Proposition 2.8.11(vii):

$$\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash t = \lfloor t \rfloor^{\omega} = \lfloor t \rfloor^{\omega} + \underline{\underline{\delta}} = t + \underline{\underline{\delta}}.$$

(iv). $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash t = \nu_{\mathrm{rel}}(t) + \sigma_{\mathrm{rel}}(s)$. Then we have, using Axiom DRT5:

$$\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash t = \nu_{\mathrm{rel}}(t) + \sigma_{\mathrm{rel}}(s) = \nu_{\mathrm{rel}}(t) + \sigma_{\mathrm{rel}}(s) + \underline{\underline{\delta}} = t + \underline{\underline{\delta}}.$$

■

Remark 2.8.15 (Representation of $\mathbf{BPA^{+}_{drt}}$ Terms)
The main use of Lemmata 2.8.13 and 2.8.14 will be in induction proofs regarding regarding the (yet to be treated) theories $\mathrm{PA}^{+}_{\mathrm{drt}}$ and $\mathrm{ACP}^{+}_{\mathrm{drt}}$ (see Sections 3.6 and 3.7).

Theorem 2.8.16 (Elimination for $\mathbf{BPA^{+}_{drt}}$)
Let t be a closed BPA_{drt} term. Then there is a basic term s such that $BPA^{+}_{drt} \vdash t = s$.

Proof First a term rewriting system is given. Then, it is shown that this term rewriting system is strongly normalizing and that the normal forms of the closed $\mathrm{BPA}_{\mathrm{drt}}$ terms are basic terms. The term rewriting system is given in Table 22. The rewriting rules

$(x + y) \cdot z \to x \cdot z + y \cdot z$	RA4
$(x \cdot y) \cdot z \to x \cdot (y \cdot z)$	RA5
$\sigma_{\mathrm{rel}}(x) \cdot y \to \sigma_{\mathrm{rel}}(x \cdot y)$	RDRT2
$\lfloor \underline{\underline{a}} \rfloor^{\omega} \to a$	RATS
$\lfloor a \rfloor^{\omega} \to a$	RUSD1
$\lfloor x \cdot y \rfloor^{\omega} \to \lfloor x \rfloor^{\omega} \cdot y$	RUSD2
$\lfloor x + y \rfloor^{\omega} \to \lfloor x \rfloor^{\omega} + \lfloor y \rfloor^{\omega}$	RUSD3
$\lfloor \sigma_{\mathrm{rel}}(x) \rfloor^{\omega} \to \delta$	RUSD4
$\lfloor \dot{\delta} \rfloor^{\omega} \to \delta$	RUSD5
$\dot{\delta} \cdot x \to \dot{\delta}$	RA7ID
$\nu_{\mathrm{rel}}(\underline{\underline{a}}) \to \underline{\underline{a}}$	RDCS1
$\nu_{\mathrm{rel}}(x + y) \to \nu_{\mathrm{rel}}(x) + \nu_{\mathrm{rel}}(y)$	RDCS2
$\nu_{\mathrm{rel}}(x \cdot y) \to \nu_{\mathrm{rel}}(x) \cdot y$	RDCS3
$\nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x)) \to \underline{\underline{\delta}}$	RDCS4
$\nu_{\mathrm{rel}}(a) \to \underline{\underline{a}}$	RDCS5
$\nu_{\mathrm{rel}}(\dot{\delta}) \to \dot{\delta}$	RDCSID

Table 22: Term rewriting system for $\mathrm{BPA}_{\mathrm{drt}}$.

RA4, RA5, RDRT2, RATS, RA7ID, RDCS1–RDCS4, and RDCSID are obtained directly from the axioms. The rewriting rules RUSD1–RUSD5 and RDCS5 are added to deal properly with the recursive definition of ultimate start delay. The corresponding equalities are derivable from the axioms as is shown in Proposition 2.8.11.

With the method of the lexicographical path ordering it is shown that the term rewriting system is strongly normalizing. Thereto the operator $\cdot$ is assigned the lexicographical status for the first argument and the following well-founded partial ordering on the signature of $\mathrm{BPA}_{\mathrm{drt}}$ is defined:

$$\begin{array}{ccccc}
 & & \lfloor\ \rfloor^{\omega} & > & a \\
 & & \vee & & \\
\nu_{\mathrm{rel}} & > & \cdot & > & + \\
\vee & & \vee & & \\
\underline{\underline{a}} & & \sigma_{\mathrm{rel}} & &
\end{array}$$

We give the following reduction for the rewriting rule RUSD5:

$$\begin{aligned} \lfloor \dot{\delta} \rfloor^{\omega} &\succ_{\mathrm{lpo}} \lfloor \dot{\delta} \rfloor^{\omega\star} \\ &\succ_{\mathrm{lpo}} \delta \end{aligned}$$

Note that the reductions for the other rewriting rules have already been given in the proofs of previous elimination theorems.

It remains to prove that every normal form of a closed $\mathrm{BPA}_{\mathrm{drt}}$ term is a basic term. Suppose that s is the normal form of a closed $\mathrm{BPA}_{\mathrm{drt}}$ term. Furthermore, suppose that s is not a basic term and that s' is the smallest subterm of s which is not a basic term. We distinguish all possible cases:

(i). s' is of the form $\underline{\underline{a}}$ or a for some $a \in A_{\delta}$, or of the form $\dot{\delta}$. Then s' is clearly a basic term, so this case does not occur.

(ii). s' is of the form $s_1 \cdot s_2$ for basic terms s_1 and s_2. With respect to basic term s_1 the following cases can be distinguished:

- (a) $s_1 \equiv \dot{\delta}$; then RA7ID is applicable, so s' is not a normal form.
- (b) $s_1 \equiv \underline{\underline{a}}$ for some $a \in A_{\delta}$. Then s' is a basic term. This contradicts the assumption that s' is not a basic term.
- (c) $s_1 \equiv a$ for some $a \in A_{\delta}$. Then s' is a basic term, and we have again a contradiction.
- (d) $s_1 \equiv \underline{\underline{a}} \cdot s'_1$ for some $a \in A_{\delta}$ and basic term s'_1. Then rewriting rule RA5 is applicable, so s' is not a normal form.
- (e) $s_1 \equiv a \cdot s'_1$ for some $a \in A_{\delta}$ and some basic term s'_1. Then rewriting rule RA5 is applicable, so s' is not a normal form.
- (f) $s_1 \equiv s'_1 + s''_1$ for some basic terms s'_1 and s''_2. Then rewriting rule RA4 is applicable, so s' is not a normal form.
- (g) $s_1 \equiv \sigma_{\mathrm{rel}}(s'_1)$ for some basic term s'_1. Then rewriting rule RDRT2 is applicable, so s' is not a normal form.

(iii). s' is of the form $s'_1 + s'_2$ for basic terms s'_1 and s'_2. Then s' is a basic term itself, so this case cannot happen.

(iv). s' is of the form $\sigma_{\mathrm{rel}}(s'')$ for some basic term s''. Then again s' is a basic term itself, so this case cannot happen either.

(v). s' is of the form $\nu_{\mathrm{rel}}(s'')$, where s'' is a basic term. Then one of RDCS1–RDCS5 or RDCSID can be applied, so s' is not a normal form.

(vi). s' is of the form $\lfloor s'' \rfloor^{\omega}$ for some basic term s''. Then one of RATS or RUSD1–RUSD5 can be applied, so s' is not a normal form.

In every case s' is a basic term or a rewriting rule is applicable. If s' is a basic term this contradicts the assumption that it is not. If a rewriting rule is applicable then s' and s are not a normal form. This contradicts the assumption that s is a normal form. From this contradiction we conclude that s is a basic term. ■

Remark 2.8.17 (Elimination for BPA$_{\mathbf{drt}}$)
Elimination for a somewhat different version of BPA$_{\mathrm{drt}}$ is also claimed (without proof) in Section 3.4 of [10].

Theorem 2.8.18 (Soundness of BPA$^{+}_{\mathbf{drt}}$)
The set of closed BPA$_{drt}$ terms modulo bisimulation equivalence is a model of BPA$^{+}_{drt}$.

Proof The soundness of each of the axioms of BPA$_{\mathrm{drt}}$ has already been proven in the previous soundness theorems, so we do not repeat those proofs here. ■

Remark 2.8.19 (Soundness of BPA$_{\mathbf{drt}}$)
Soundness of a somewhat different version of BPA$_{\mathrm{drt}}$ is also claimed (without proof) in Section 3.5 of [10].

Lemma 2.8.20 (Towards Completeness of BPA$^{+}_{\mathbf{drt}}$)
Let x be a closed BPA$_{drt}$ term and let $a \in A$. Then we have:

(i). $T(BPA_{drt}) \models x \xrightarrow{a} \surd \implies BPA^{+}_{drt} \vdash x = \underline{\underline{a}} + x,$

(ii). $T(BPA_{drt}) \models x \xrightarrow{a} y \implies BPA^{+}_{drt} \vdash x = \underline{\underline{a}} \cdot y + x,$

(iii). $T(BPA_{drt}) \models ID(x) \implies BPA^{+}_{drt} \vdash x = \dot{\delta},$

(iv). $T(BPA_{drt}) \models \neg ID(x) \implies BPA^{+}_{drt} \vdash x + \underline{\underline{\delta}} = x,$

(v). $T(BPA_{drt}) \models x \overset{\sigma}{\nrightarrow} \implies BPA^{+}_{drt} \vdash x = \nu_{rel}(x),$

(vi). $T(BPA_{drt}) \models x \xrightarrow{\sigma} y \implies BPA^{+}_{drt} \vdash x = \sigma_{rel}(y) + \nu_{rel}(x),$

(vii). $T(BPA_{drt}) \models x \xrightarrow{\sigma} x \implies BPA^{+}_{drt} \vdash x = \lfloor x \rfloor^{\omega},$

(viii). $T(BPA_{drt}) \models x \xrightarrow{a} y \implies n(x) > n(y),$

(ix). $T(BPA_{drt}) \models x \xrightarrow{\sigma} y \implies x \equiv y \vee n(x) > n(y).$

Proof For part (i)–(vii) we assume, by Theorem 2.8.16 and Theorem 2.8.18, without loss of generality, that x is a basic term, and then apply induction on the structure of basic terms. For part (viii) and (ix) we again have to use induction on the general structure of terms.

(i). Suppose that $T(\mathrm{BPA_{drt}}) \models x \xrightarrow{a} \surd$. Case 1: $x \equiv \dot{\delta}$. This is in contradiction with $T(\mathrm{BPA_{drt}}) \models x \xrightarrow{a} \surd$, so this case does not occur. Case 2: $x \equiv \underline{b}$, where $b \in A_{\delta}$. Because $T(\mathrm{BPA_{drt}}) \models x \xrightarrow{a} \surd$, it must be the case that $b \equiv a$. So we have $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = \underline{\underline{b}} = \underline{\underline{b}} + \underline{\underline{b}} = \underline{\underline{a}} + \underline{\underline{b}} = \underline{\underline{a}} + x$. Case 3: $x \equiv b$, where $b \in A_{\delta}$. Because $T(\mathrm{BPA_{drt}}) \models x \xrightarrow{a} \surd$, it must be the case that $b \equiv a$. So we have $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = b =$

$\lfloor \underline{\underline{b}} \rfloor^{\omega} = \nu_{\mathrm{rel}}(\underline{\underline{b}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{b}} \rfloor^{\omega}) = \nu_{\mathrm{rel}}(\underline{\underline{b}}) + \nu_{\mathrm{rel}}(\underline{\underline{b}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{b}} \rfloor^{\omega}) = \underline{\underline{b}} + \lfloor \underline{\underline{b}} \rfloor^{\omega} = \underline{\underline{a}} + b = \underline{\underline{a}} + x$. Case 4: $x \equiv \underline{\underline{b}} \cdot x'$, where $b \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} \surd$, so this case does not occur. Case 5: $x \equiv b \cdot x'$, where $b \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} \surd$, so this case does not occur. Case 6: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} \surd$, necessarily $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \xrightarrow{a} \surd$ or $T(\mathrm{BPA}_{\mathrm{drt}}) \models x'' \xrightarrow{a} \surd$. Therefore, by the induction hypothesis, $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x' = \underline{\underline{a}} + x'$ or $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x'' = \underline{\underline{a}} + x''$. But then in both cases $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = x' + x'' = \underline{\underline{a}} + x' + x'' = \underline{\underline{a}} + x$. Case 7: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} \surd$, so this case does not occur.

(ii). Suppose that $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$. Case 1: $x \equiv \dot{\delta}$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, so this case does not occur. Case 2: $x \equiv \underline{\underline{b}}$, where $b \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, so this case does not occur. Case 3: $x \equiv b$, where $b \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, so this case does not occur. Case 4: $x \equiv \underline{\underline{b}} \cdot x'$, where $b \in A_\delta$ and x' is a basic term. Then, because $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, it must be that $b \equiv a$ and $x' \equiv y$. So, $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = x + x = \underline{\underline{b}} \cdot x' + x = \underline{\underline{a}} \cdot y + x$. Case 5: $x \equiv b \cdot x'$, where $b \in A_\delta$ and x' is a basic term. Then, because $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, it must be that $b \equiv a$ and $x' \equiv y$. So, $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = b \cdot x' = a \cdot y = (\underline{\underline{a}} + a) \cdot y = \underline{\underline{a}} \cdot y + a \cdot y = \underline{\underline{a}} \cdot y + x$. Case 6: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, necessarily $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \xrightarrow{a} y$ or $T(\mathrm{BPA}_{\mathrm{drt}}) \models x'' \xrightarrow{a} y$. Therefore, by the induction hypothesis, $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x' = \underline{\underline{a}} \cdot y + x'$ or $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x'' = \underline{\underline{a}} \cdot y + x''$. But then in both cases $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = x' + x'' = \underline{\underline{a}} \cdot y + x' + x'' = \underline{\underline{a}} \cdot y + x$. Case 7: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, so this case does not occur.

(iii). Suppose that $T(\mathrm{BPA}_{\mathrm{drt}}) \models \mathrm{ID}(x)$. Case 1: $x \equiv \dot{\delta}$. Then $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = \dot{\delta}$ is trivially fulfilled. Case 2: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models \mathrm{ID}(x)$, so this case does not occur. Case 3: $x \equiv a$, where $a \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models \mathrm{ID}(x)$, so this case does not occur. Case 4: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models \mathrm{ID}(x)$, so this case does not occur. Case 5: $x \equiv a \cdot x'$, where $a \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models \mathrm{ID}(x)$, so this case does not occur. Case 6: $x \equiv x' + x''$, where x' and x'' are basic terms. Then, because $T(\mathrm{BPA}_{\mathrm{drt}}) \models \mathrm{ID}(x)$, it must be the case that $T(\mathrm{BPA}_{\mathrm{drt}}) \models \mathrm{ID}(x'), \mathrm{ID}(x'')$. So, by the induction hypothesis, we have that $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x' = \dot{\delta}, x'' = \dot{\delta}$. But then also $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = x' + x'' = \dot{\delta} + \dot{\delta} = \dot{\delta}$. Case 7: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models \mathrm{ID}(x)$, so this case does not occur.

(iv). Suppose that $T(\mathrm{BPA}_{\mathrm{drt}}) \models \neg\mathrm{ID}(x)$. Case 1: $x \equiv \dot{\delta}$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models \neg\mathrm{ID}(x)$, so this case does not occur. Case 2: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x + \underline{\underline{\delta}} = \underline{\underline{a}} + \underline{\underline{\delta}} = \underline{\underline{a}} = x$. Case 3: $x \equiv a$, where $a \in A_\delta$. Then we have $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x + \underline{\underline{\delta}} = a + \underline{\underline{\delta}} = \lfloor \underline{\underline{a}} \rfloor^{\omega} + \underline{\underline{\delta}} = \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega}) + \underline{\underline{\delta}} = \underline{\underline{a}} + \underline{\underline{\delta}} + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega}) = \underline{\underline{a}} + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega}) = \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega}) = \lfloor \underline{\underline{a}} \rfloor^{\omega} = a = x$. Case 4: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x + \underline{\underline{\delta}} = \underline{\underline{a}} \cdot x' + \underline{\underline{\delta}} = \underline{\underline{a}} \cdot x' + \underline{\underline{\delta}} \cdot x' = (\underline{\underline{a}} + \underline{\underline{\delta}}) \cdot x' = \underline{\underline{a}} \cdot x' = x$. Case 5: $x \equiv a \cdot x'$, where $a \in A_\delta$ and x' is a basic term. Then, using Case 3, $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x + \underline{\underline{\delta}} = a \cdot x' + \underline{\underline{\delta}} = a \cdot x' + \underline{\underline{\delta}} \cdot x' = (a + \underline{\underline{\delta}}) \cdot x' = a \cdot x' = x$. Case 6: $x \equiv x' + x''$, where x'

and x'' are basic terms. As $T(\mathrm{BPA}_{\mathrm{drt}}) \models \neg\mathrm{ID}(x)$, necessarily $T(\mathrm{BPA}_{\mathrm{drt}}) \models \neg\mathrm{ID}(x')$ or $T(\mathrm{BPA}_{\mathrm{drt}}) \models \neg\mathrm{ID}(x'')$. Therefore, by the induction hypothesis, $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x' + \underline{\underline{\delta}} = x'$ or $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x'' + \underline{\underline{\delta}} = x''$. So, in both cases, $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x + \underline{\underline{\delta}} = x' + x'' + \underline{\underline{\delta}} = x' + x'' = x$. Case 7: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. Then we have $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(x') + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(x') = x$.

(v). Suppose that $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \overset{\sigma}{\nrightarrow}$. Case 1: $x \equiv \dot{\delta}$. By Axiom DCSID we have $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = \dot{\delta} = \nu_{\mathrm{rel}}(\dot{\delta}) = \nu_{\mathrm{rel}}(x)$. Case 2: $x \equiv \underline{\underline{a}}$, where $a \in A_{\delta}$. We have $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = \underline{\underline{a}} = \nu_{\mathrm{rel}}(\underline{\underline{a}}) = \nu_{\mathrm{rel}}(x)$. Case 3: $x \equiv a$, where $a \in A_{\delta}$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \overset{\sigma}{\nrightarrow}$, so this case does not occur. Case 4: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_{\delta}$ and x' is a basic term. We have $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = \underline{\underline{a}} \cdot x' = \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot x' = \nu_{\mathrm{rel}}(\underline{\underline{a}} \cdot x') = \nu_{\mathrm{rel}}(x)$. Case 5: $x \equiv a \cdot x'$, where $a \in A_{\delta}$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \overset{\sigma}{\nrightarrow}$, so this case does not occur. Case 6: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \overset{\sigma}{\nrightarrow}$, necessarily $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \overset{\sigma}{\nrightarrow}$ and $T(\mathrm{BPA}_{\mathrm{drt}}) \models x'' \overset{\sigma}{\nrightarrow}$. Therefore, by the induction hypothesis, $\mathrm{BPA}_{\mathrm{drt}} \vdash x' = \nu_{\mathrm{rel}}(x')$ and $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x'' = \nu_{\mathrm{rel}}(x'')$. But then also $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = x' + x'' = \nu_{\mathrm{rel}}(x') + \nu_{\mathrm{rel}}(x'') = \nu_{\mathrm{rel}}(x' + x'') = \nu_{\mathrm{rel}}(x)$. Case 7: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. Then, by $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \overset{\sigma}{\nrightarrow}$, it must be the case that $\mathrm{ID}(x')$. So, by (iii), we have that $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x' = \dot{\delta}$. Therefore, $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = \sigma_{\mathrm{rel}}(x') = \sigma_{\mathrm{rel}}(\dot{\delta}) = \underline{\underline{\delta}} = \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x')) = \nu_{\mathrm{rel}}(x)$.

(vi). Suppose that $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$. Case 1: $x \equiv \dot{\delta}$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 2: $x \equiv \underline{\underline{a}}$, where $a \in A_{\delta}$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 3: $x \equiv a$, where $a \in A_{\delta}$. Because $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, it must be the case that $y \equiv a$. So we have $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = a = \lfloor \underline{\underline{a}} \rfloor^{\omega} = \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) = \sigma_{\mathrm{rel}}(a) + \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(a) + \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \nu_{\mathrm{rel}}\left(\sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) = \sigma_{\mathrm{rel}}(a) + \nu_{\mathrm{rel}}\left(\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) = \sigma_{\mathrm{rel}}(a) + \nu_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) = \sigma_{\mathrm{rel}}(a) + \nu_{\mathrm{rel}}(a) = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$. Case 4: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_{\delta}$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 5: $x \equiv a \cdot x'$, where $a \in A_{\delta}$ and x' is a basic term. Because $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, it must be the case that $y \equiv a \cdot x'$. So we have $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = a \cdot x' = \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot x' = \left(\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) \cdot x' = \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot x' + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) \cdot x' = \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) \cdot x' + \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot x' + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(a \cdot x') + \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot x' + \underline{\underline{\delta}} \cdot x' = \sigma_{\mathrm{rel}}(y) + (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \underline{\underline{\delta}}) \cdot x' = \sigma_{\mathrm{rel}}(y) + \left(\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \nu_{\mathrm{rel}}\left(\sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right)\right) \cdot x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}\left(\underline{\underline{a}} + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) \cdot x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}\left(\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right)\right) \cdot x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}\left(\lfloor \underline{\underline{a}} \rfloor^{\omega}\right) \cdot x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(a) \cdot x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(a \cdot x') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$. Case 6: $x \equiv x' + x''$, where x' and x'' are basic terms. As $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, necessarily (1) $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \xrightarrow{\sigma} y, x'' \overset{\sigma}{\nrightarrow}$, or, (2) $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \overset{\sigma}{\nrightarrow}, x'' \xrightarrow{\sigma} y$, or, (3) $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \xrightarrow{\sigma} y', x'' \xrightarrow{\sigma} y''$ where $y \equiv y' + y''$. In the first case, by the induction hypothesis, we have $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x')$, and, by (v), $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x'' = \nu_{\mathrm{rel}}(x'')$. Therefore, $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = x' + x'' = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x') + \nu_{\mathrm{rel}}(x'') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x' + x'') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$. The second case is treated analogously. In the third case we have, by the induction hypothesis, $\mathrm{BPA}_{\mathrm{drt}} \vdash x' = \sigma_{\mathrm{rel}}(y') + \nu_{\mathrm{rel}}(x'), x'' = \sigma_{\mathrm{rel}}(y'') + \nu_{\mathrm{rel}}(x'')$. Therefore we have $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = x' + x'' = \sigma_{\mathrm{rel}}(y') + \nu_{\mathrm{rel}}(x') + \sigma_{\mathrm{rel}}(y'') + \nu_{\mathrm{rel}}(x'') = \sigma_{\mathrm{rel}}(y' + y'') + \nu_{\mathrm{rel}}(x' + x'') = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$. Case 7: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. Because $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, it must be the case that $x' \equiv y$. So we have $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = \sigma_{\mathrm{rel}}(x') = \sigma_{\mathrm{rel}}(y) = \sigma_{\mathrm{rel}}(y) + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x')) = \sigma_{\mathrm{rel}}(y) + \nu_{\mathrm{rel}}(x)$.

(vii). Suppose that $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} x$. Case 1: $x \equiv \dot{\delta}$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} x$, so this case does not occur. Case 2: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} x$, so this case does not occur. Case 3: $x \equiv a$, where $a \in A_\delta$. Then we have, using Proposition 2.8.11(i), $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = a = \lfloor a \rfloor^{\omega} = \lfloor x \rfloor^{\omega}$. Case 4: $x \equiv \underline{\underline{a}} \cdot x'$, where $a \in A_\delta$ and x' is a basic term. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} x$, so this case does not occur. Case 5: $x \equiv a \cdot x'$, where $a \in A_\delta$ and x' is a basic term. Then we can derive, using Proposition 2.8.11(i) and (ii), $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = a \cdot x' = \lfloor a \rfloor^{\omega} \cdot x' = \lfloor a \cdot x' \rfloor^{\omega} = \lfloor x \rfloor^{\omega}$. Case 6: $x \equiv x' + x''$, where x' and x'' are basic terms. Then we can derive, using Proposition 2.8.11(iii) and the induction hypothesis, $\mathrm{BPA}^{+}_{\mathrm{drt}} \vdash x = x' + x'' = \lfloor x \rfloor^{\omega} + \lfloor x' \rfloor^{\omega} = \lfloor x' + x'' \rfloor^{\omega} = \lfloor x \rfloor^{\omega}$. Case 7: $x \equiv \sigma_{\mathrm{rel}}(x')$, where x' is a basic term. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} x$, so this case does not occur.

(viii). Suppose that $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$. Case 1: $x \equiv \dot{\delta}$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 2: $x \equiv \underline{\underline{b}}$, where $b \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, so this case does not occur. Case 3: $x \equiv b$, where $b \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, so this case does not occur. Case 4: $x \equiv x' \cdot x''$, for certain terms x' and x''. Then, because $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, we either have $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \xrightarrow{a} \surd$ and $y \equiv x''$, or we have $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \xrightarrow{a} x'''$ and $y \equiv x''' \cdot x''$ for some term x'''. In the first case, we have $n(x) = n(x' \cdot x'') = n(x') + n(x'') + 1 > n(x'') = n(y)$, and in the second we can apply the induction hypothesis to arrive at $n(x') > n(x''')$, so we get $n(x) = n(x' \cdot x'') = n(x') + n(x'') + 1 > n(x''') + n(x') + 1 = n(x''' \cdot x'') = n(y)$. Case 5: $x \equiv x' + x''$, for certain terms x' and x''. Since $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, necessarily $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \xrightarrow{a} y$ or $T(\mathrm{BPA}_{\mathrm{drt}}) \models x'' \xrightarrow{a} y$. Therefore, by the induction hypothesis, $n(x') > n(y)$ or $n(x'') > n(y)$. In both cases $n(x) = n(x' + x'') = n(x') + n(x'') + 1 > n(y)$. Case 6: $x \equiv \sigma_{\mathrm{rel}}(x')$, for a certain term x'. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, so this case does not occur. Case 7: $x \equiv \nu_{\mathrm{rel}}(x')$, for a certain term x'. Since $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, necessarily $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \xrightarrow{a} y$. Therefore, by the induction hypothesis, $n(x') > n(y)$. So, $n(x) = n(\nu_{\mathrm{rel}}(x')) = n(x') + 1 > n(y)$. Case 8: $x \equiv \lfloor x' \rfloor^{\omega}$, for a certain term x'. Since $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{a} y$, necessarily $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \xrightarrow{a} y$. Therefore, by the induction hypothesis, $n(x') > n(y)$. So, $n(x) = n(\lfloor x' \rfloor^{\omega}) = n(x') + 1 > n(y)$.

(ix). Suppose that $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$. Case 1: $x \equiv \dot{\delta}$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 2: $x \equiv \underline{\underline{a}}$, where $a \in A_\delta$. This is in contradiction with $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 3: $x \equiv a$, where $a \in A_\delta$. Because $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, it must be the case that $x \equiv y$, and we are done. Case 4: $x \equiv x' \cdot x''$, for certain terms x' and x''. Then necessarily, $x' \xrightarrow{\sigma} x'''$ and $y \equiv x''' \cdot x''$ for some term x'''. We now can apply the induction hypothesis to arrive at $n(x') > n(x''')$, so we get $n(x) = n(x' \cdot x'') = n(x') + n(x'') + 1 > n(x''') + n(x'') + 1 = n(x''' \cdot x'') = n(y)$. Case 5: $x \equiv x' + x''$, for certain terms x' and x''. As $T(\mathrm{BPA}_{\mathrm{drt}}) \models x \xrightarrow{\sigma} y$, necessarily (1) $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \xrightarrow{\sigma} y, x'' \stackrel{\sigma}{\nrightarrow}$, or, (2) $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \stackrel{\sigma}{\nrightarrow}, x'' \xrightarrow{\sigma} y$, or, (3) $T(\mathrm{BPA}_{\mathrm{drt}}) \models x' \xrightarrow{\sigma} y', x'' \xrightarrow{\sigma} y''$ where $y \equiv y' + y''$. In the first case, by the induction hypothesis, $n(x') > n(y)$. So $n(x) = n(x' + x'') = n(x') + n(x'') + 1 > n(y)$. The second case is treated analogously. In the third case, by the induction hypothesis, $n(x') > n(y')$ and $n(x'') > n(y'')$. So $n(x) =$

$n(x'+x'') = n(x')+n(x'')+1 > n(y')+n(y'')+1 = n(y)$. Case 6: $x \equiv \sigma_{\rm rel}(x')$, for a certain term x'. Because $T(\mathrm{BPA_{drt}}) \models x \xrightarrow{\sigma} y$, it must be the case that $x' \equiv y$. Then we have $n(x) = n(\sigma_{\rm rel}(x')) = n(x')+1 = n(y)+1 > n(y)$. Case 7: $x \equiv \nu_{\rm rel}(x')$, for a certain term x'. This is in contradiction with $T(\mathrm{BPA_{drt}}) \models x \xrightarrow{\sigma} y$, so this case does not occur. Case 8: $x \equiv \lfloor x' \rfloor^{\omega}$, for a certain term x'. Because $T(\mathrm{BPA_{drt}}) \models x \xrightarrow{\sigma} y$, it must be the case that $x \equiv y$, and we are done.

■

Remark 2.8.21 (Towards Completeness of $\mathbf{BPA^{+}_{drt}}$)
Note that Lemma 2.8.14 on page 59 now also follows as a corollary from Lemma 2.8.20(iii) and (iv) by the law of the excluded middle.

Theorem 2.8.22 (Completeness of $\mathbf{BPA^{+}_{drt}}$)
The axiom system BPA^{+}_{drt} is a complete axiomatization of the set of closed BPA_{drt} terms modulo bisimulation equivalence.

Proof Suppose that $s + t \sim_{\mathrm{BPA_{drt}}} t$. We will prove that $\mathrm{BPA^{+}_{drt}} \vdash s + t = t$. By Theorem 2.8.16 we can restrict ourselves to basic terms s and t. The proof is done with induction on $n(s)$, using Lemma 2.8.20(viii)–(ix) and case distinction on the form of basic term s.

(i). $s \equiv \dot{\delta}$. Using Axiom A6ID we have $\mathrm{BPA^{+}_{drt}} \vdash s + t = \dot{\delta} + t = t + \dot{\delta} = t$.

(ii). $s \equiv \underline{\underline{\delta}}$. Then, since for $u \in A_{\sigma}$ we have $T(\mathrm{BPA_{drt}}) \models s \overset{u}{\nrightarrow}$, also $T(\mathrm{BPA_{drt}}) \models s + t \xrightarrow{u} p$ iff $T(\mathrm{BPA_{drt}}) \models t \xrightarrow{u} p$, and for $a \in A$, $T(\mathrm{BPA_{drt}}) \models s + t \xrightarrow{a} \surd$ iff $T(\mathrm{BPA_{drt}}) \models t \xrightarrow{a} \surd$. Furthermore, we have $\neg\mathrm{ID}(s+t)$, since $T(\mathrm{BPA_{drt}}) \models \neg\mathrm{ID}(s)$. Since $s + t \sim_{\mathrm{BPA_{drt}}} t$, we also have $T(\mathrm{BPA_{drt}}) \models \neg\mathrm{ID}(t)$. Using Lemma 2.8.20(iv) we have $\mathrm{BPA^{+}_{drt}} \vdash s + t = \underline{\underline{\delta}} + t = t + \underline{\underline{\delta}} = t$.

(iii). $s \equiv \underline{\underline{a}}$, where $a \in A$. From the deduction rules we have $T(\mathrm{BPA_{drt}}) \models s \xrightarrow{a} \surd$ and $T(\mathrm{BPA_{drt}}) \models s+t \xrightarrow{a} \surd$. Since $s+t \sim_{\mathrm{BPA_{drt}}} t$ we also have $T(\mathrm{BPA_{drt}}) \models t \xrightarrow{a} \surd$. By Lemma 2.8.20(i) we obtain $\mathrm{BPA^{+}_{drt}} \vdash t = \underline{\underline{a}} + t$. So, $\mathrm{BPA^{+}_{drt}} \vdash s + t = \underline{\underline{a}} + t = t$.

(iv). $s \equiv \delta$. Then $\delta \xrightarrow{\sigma} \delta$. Therefore $s+t \xrightarrow{\sigma} s+t'$ and $t \xrightarrow{\sigma} t'$ with $s+t' \sim_{\mathrm{BPA_{drt}}} t'$. With Lemma 2.8.20(vi) we have $\mathrm{BPA^{+}_{drt}} \vdash t = \sigma_{\rm rel}(t') + \nu_{\rm rel}(t)$. Two cases need to be considered:

  (a) $t \equiv t'$. Now, $s + t \xrightarrow{\sigma} s + t$ and $t \xrightarrow{\sigma} t$, so by Lemma 2.8.20(vii) we have $\mathrm{BPA^{+}_{drt}} \vdash s + t = \lfloor s + t \rfloor^{\omega}$ and $\mathrm{BPA^{+}_{drt}} \vdash t = \lfloor t \rfloor^{\omega}$. So we can derive, using Proposition 2.8.11(i) and (iii) and Lemma 2.8.20(iv): $\mathrm{BPA^{+}_{drt}} \vdash s+t = \lfloor s + t \rfloor^{\omega} = \lfloor s \rfloor^{\omega} + \lfloor t \rfloor^{\omega} = \lfloor \delta \rfloor^{\omega} + \lfloor t \rfloor^{\omega} = \delta + \lfloor t \rfloor^{\omega} = \lfloor \underline{\underline{\delta}} \rfloor^{\omega} + \lfloor t \rfloor^{\omega} = \lfloor \underline{\underline{\delta}} + t \rfloor^{\omega} = \lfloor t \rfloor^{\omega} = t$.

  (b) $t \not\equiv t'$. Now, by Lemma 2.8.20(ix), $n(t') < n(t)$. Therefore, the induction hypothesis is applicable: $\mathrm{BPA^{+}_{drt}} \vdash \delta + t' = t'$. Consider the following computation: $\mathrm{BPA^{+}_{drt}} \vdash s + t = \delta + t = \lfloor \underline{\underline{\delta}} \rfloor^{\omega} + t = \nu_{\rm rel}(\underline{\underline{\delta}}) + \sigma_{\rm rel}(\lfloor \underline{\underline{\delta}} \rfloor^{\omega}) + t = \underline{\underline{\delta}} + \sigma_{\rm rel}(\delta) + t = \sigma_{\rm rel}(\delta) + t = \sigma_{\rm rel}(\delta) + \sigma_{\rm rel}(t') + \nu_{\rm rel}(t) = \sigma_{\rm rel}(\delta + t') + \nu_{\rm rel}(t) = \sigma_{\rm rel}(t') + \nu_{\rm rel}(t) = t$.

(v). $s \equiv a$, where $a \in A$. Then $s \xrightarrow{a} \surd$. Therefore $s + t \xrightarrow{a} \surd$ and, since $s + t \sim_{\mathrm{BPA_{drt}}} t$, $t \xrightarrow{a} \surd$. Using Lemma 2.8.20(i) we obtain $\mathrm{BPA^{+}_{drt}} \vdash t = \underline{\underline{a}} + t$. We also have $s \xrightarrow{\sigma} s$. Therefore $s + t \xrightarrow{\sigma} s + t'$ and $t \xrightarrow{\sigma} t'$. From Lemma 2.8.20(vi) we obtain: $\mathrm{BPA^{+}_{drt}} \vdash t = \sigma_{\rm rel}(t') + \nu_{\rm rel}(t)$. Two cases can be distinguished:

(a) $t \equiv t'$. Now, $s + t \xrightarrow{\sigma} s + t$ and $t \xrightarrow{\sigma} t$, so by Lemma 2.8.20(vii) we have $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s + t = \lfloor s + t \rfloor^\omega$ and $\mathrm{BPA}^+_{\mathrm{drt}} \vdash t = \lfloor t \rfloor^\omega$. So we can derive, using Proposition 2.8.11(i) and (iii): $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s + t = \lfloor s + t \rfloor^\omega = \lfloor s \rfloor^\omega + \lfloor t \rfloor^\omega = \lfloor a \rfloor^\omega + \lfloor t \rfloor^\omega = a + \lfloor t \rfloor^\omega = \lfloor \underline{\underline{a}} \rfloor^\omega + \lfloor t \rfloor^\omega = \lfloor \underline{\underline{a}} + t \rfloor^\omega = \lfloor t \rfloor^\omega = t$.

(b) $t \not\equiv t'$. Now, by Lemma 2.8.20(ix), $n(t') < n(t)$. Therefore the induction hypothesis is applicable: $\mathrm{BPA}^+_{\mathrm{drt}} \vdash a + t' = t'$. Consider the following computation: $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s + t = a + t = \lfloor \underline{\underline{a}} \rfloor^\omega + t = \nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^\omega) + t = \underline{\underline{a}} + \sigma_{\mathrm{rel}}(a) + t = \sigma_{\mathrm{rel}}(a) + t = \sigma_{\mathrm{rel}}(a) + \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(a + t') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t) = t$.

(vi). $s \equiv \underline{\underline{\delta}} \cdot s'$, where s' is a basic term. Then we have $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s = \underline{\underline{\delta}} \cdot s' = \underline{\underline{\delta}}$ and, using (ii), $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s + t = t$.

(vii). $s \equiv \underline{\underline{a}} \cdot s'$, where $a \in A$ and s' is a basic term. From the deduction rules we obtain $T(\mathrm{BPA}_{\mathrm{drt}}) \models s \xrightarrow{a} s'$ and $T(\mathrm{BPA}_{\mathrm{drt}}) \models s + t \xrightarrow{a} s'$. Since $s + t \sim_{\mathrm{BPA}_{\mathrm{drt}}} t$, we then also have $T(\mathrm{BPA}_{\mathrm{drt}}) \models t \xrightarrow{a} t'$ for some t' such that $s' \sim_{\mathrm{BPA}_{\mathrm{drt}}} t'$. By the induction hypothesis we have $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s' = t'$. From Lemma 2.8.20(ii) we have $\mathrm{BPA}^+_{\mathrm{drt}} \vdash t = \underline{\underline{a}} \cdot t' + t$. So, $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s + t = \underline{\underline{a}} \cdot s' + t = \underline{\underline{a}} \cdot t' + t = t$.

(viii). $s \equiv \delta \cdot s'$, where s' is a basic term. Then we have, using Proposition 2.8.12, $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s = \delta \cdot s' = \delta$ and, using (iv), $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s + t = t$.

(ix). $s \equiv a \cdot s'$, where $a \in A$ and s' is a basic term. Then $s \xrightarrow{a} s'$ and $s + t \xrightarrow{a} s'$. Since $s + t \sim_{\mathrm{BPA}_{\mathrm{drt}}} t$ we also have $t \xrightarrow{a} t'$ for some t' such that $s' \sim_{\mathrm{BPA}_{\mathrm{drt}}} t'$. By induction we therefore have $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s' = t'$. We also have $s \xrightarrow{\sigma} s$ and $s + t \xrightarrow{\sigma} s + t''$ and $t \xrightarrow{\sigma} t''$. By Lemma 2.8.20(ii) we have $\mathrm{BPA}^+_{\mathrm{drt}} \vdash t = \underline{\underline{a}} \cdot t' + t$ and $\mathrm{BPA}^+_{\mathrm{drt}} \vdash t = \sigma_{\mathrm{rel}}(t'') + \nu_{\mathrm{rel}}(t)$. Two cases can be distinguished:

(a) $t \equiv t''$. Now, $s + t \xrightarrow{\sigma} s + t$ and $t \xrightarrow{\sigma} t$, so by Lemma 2.8.20(vii) we have $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s + t = \lfloor s + t \rfloor^\omega$ and $\mathrm{BPA}^+_{\mathrm{drt}} \vdash t = \lfloor t \rfloor^\omega$. So we can derive, using Proposition 2.8.11(i)–(iii): $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s + t = \lfloor s + t \rfloor^\omega = \lfloor s \rfloor^\omega + \lfloor t \rfloor^\omega = \lfloor a \cdot s' \rfloor^\omega + \lfloor t \rfloor^\omega = \lfloor a \rfloor^\omega \cdot s' + \lfloor t \rfloor^\omega = a \cdot s' + \lfloor t \rfloor^\omega = \lfloor \underline{\underline{a}} \rfloor^\omega \cdot s' + \lfloor t \rfloor^\omega = \lfloor \underline{\underline{a}} \cdot s' \rfloor^\omega + \lfloor t \rfloor^\omega = \lfloor \underline{\underline{a}} \cdot s' + t \rfloor^\omega = \lfloor \underline{\underline{a}} \cdot t' + t \rfloor^\omega = \lfloor t \rfloor^\omega = t$.

(b) $t \not\equiv t''$. Now, by Lemma 2.8.20(ix), $n(t'') < n(t)$. By the induction hypothesis we then have $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s + t'' = t''$. Consider the following computation: $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s + t = a \cdot s' + t = \lfloor \underline{\underline{a}} \rfloor^\omega \cdot s' + t = (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^\omega)) \cdot s' + t = (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \cdot s' + t = \underline{\underline{a}} \cdot t' + \sigma_{\mathrm{rel}}(a) \cdot s' + t = \sigma_{\mathrm{rel}}(a) \cdot s' + t = \sigma_{\mathrm{rel}}(a \cdot s') + t = \sigma_{\mathrm{rel}}(s) + t = \sigma_{\mathrm{rel}}(s) + \sigma_{\mathrm{rel}}(t'') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(s + t'') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(t'') + \nu_{\mathrm{rel}}(t) = t$.

(x). $s \equiv s' + s''$, where s' and s'' are basic terms. Since $s' + s'' + t \sim_{\mathrm{BPA}_{\mathrm{drt}}} t$, we also have $s' + t \sim_{\mathrm{BPA}_{\mathrm{drt}}} t$ and $s'' + t \sim_{\mathrm{BPA}_{\mathrm{drt}}} t$. By the induction hypothesis we then have $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s' + t = t, s'' + t = t$. So, $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s + t = s' + s'' + t = s' + t = t$.

(xi). $s \equiv \sigma_{\mathrm{rel}}(s')$, where s' is a basic term. From the deduction rules we have $T(\mathrm{BPA}_{\mathrm{drt}}) \models \sigma_{\mathrm{rel}}(s') \xrightarrow{\sigma} s'$ and since $s + t \sim_{\mathrm{BPA}_{\mathrm{drt}}} t$ we also have $T(\mathrm{BPA}_{\mathrm{drt}}) \models t \xrightarrow{\sigma} t', s + t \xrightarrow{\sigma} s' + t'$ for some t' such that $s' + t' \sim_{\mathrm{BPA}_{\mathrm{drt}}} t'$. By Lemma 2.8.20(vi) we have $\mathrm{BPA}^+_{\mathrm{drt}} \vdash t = \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t)$. By the induction hypothesis we have $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s' + t' = t'$. So, $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s + t = \sigma_{\mathrm{rel}}(s') + t = \sigma_{\mathrm{rel}}(s') + \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(s' + t') + \nu_{\mathrm{rel}}(t) = \sigma_{\mathrm{rel}}(t') + \nu_{\mathrm{rel}}(t) = t$.

∎

Remark 2.8.23 (Completeness of $\mathrm{BPA_{drt}}$)
Completeness of a somewhat different version of $\mathrm{BPA_{drt}}$ is also claimed (without proof) in Section 3.5 of [10].

Definition 2.8.24 (Axiom for the Ultimate Start Delay and Immediate Deadlock)
We define Axiom USD5 for the ultimate start delay of immediate deadlock as shown in Table 23. Note that it precisely corresponds to the equality of Proposition 2.8.11(v).

$$\lfloor \dot{\delta} \rfloor^{\omega} = \delta \qquad \text{USD5}$$

Table 23: Axiom for $\lfloor \dot{\delta} \rfloor^{\omega}$.

Corollary 2.8.25 (Soundness of $\mathrm{BPA_{drt}}$ + USD1–USD5)
The set of closed $\mathit{BPA_{drt}}$ terms modulo bisimulation equivalence is a model of $\mathit{BPA_{drt}}$ + USD1–USD5.

Proof This follows directly from the soundness of $\mathrm{BPA^{+}_{drt}}$ (see Theorem 2.8.18 on page 62) combined with the fact that Axioms USD1–USD5 are derivable in $\mathrm{BPA^{+}_{drt}}$ (see Proposition 2.8.11 on page 55). ∎

Corollary 2.8.26 (Completeness of $\mathrm{BPA_{drt}}$ + USD1–USD5)
If we add Axioms USD1–USD4 of Table 18 on page 45 and Axiom USD5 of Table 23 to $\mathit{BPA_{drt}}$, we again have a complete axiomatization of the set of closed $\mathit{BPA_{drt}}$ terms modulo bisimulation equivalence.

Proof Careful inspection of the dependencies between the proofs in this section reveals that the proof of Theorem 2.8.22 only relies upon RSP(USD) to ensure Proposition 2.8.11(i)–(v). So, we obviously do not need RSP(USD) anymore if we add the corresponding Axioms USD1–USD5. Note that in this way we get a purely equational axiomatization (i.e. without conditional axioms or principles). ∎

3 Concurrent Process Algebras

3.1 Introduction

In this section we prove soundness and completeness for some discrete-time concurrent process algebras, i.e. process algebras containing a merge operator. We will not explore the field of concurrent process algebras as thoroughly as we did for basic process algebra. However, we feel that we have shown which paths to take in proving soundness and completeness for concurrent process algebras.

The first algebra we examine is $\mathrm{PA}^-_{\mathrm{drt}}$–ID, which is basically $\mathrm{BPA}^-_{\mathrm{drt}}$–ID with a (free) merge operator added. We also look at $\mathrm{PA}^-_{\mathrm{drt}}$–ID′ which intuitively very similar to $\mathrm{PA}^-_{\mathrm{drt}}$–ID, but only defined in a slightly different way. We then proceed by replacing the (free) merge operator with a merge operator capable of communication; this yields $\mathrm{ACP}^-_{\mathrm{drt}}$–ID. We again examine a slightly different version: $\mathrm{ACP}^-_{\mathrm{drt}}$–ID′.

Finally, we take a look at $\mathrm{PA}_{\mathrm{drt}}$ and $\mathrm{ACP}_{\mathrm{drt}}$. These theories also contain the ultimate start delay and the immediate deadlock. As $\mathrm{ACP}_{\mathrm{drt}}$ combines all features described here in one theory, the completeness result for $\mathrm{ACP}_{\mathrm{drt}}$ is in a sense “the mother of all completeness results” when it comes to discrete-time process algebra’s.

3.2 Soundness and Completeness of $\mathrm{PA}^-_{\mathrm{drt}}$–ID

Definition 3.2.1 (Signature of $\mathrm{PA}^-_{\mathrm{drt}}$–ID)
The signature of $\mathrm{PA}^-_{\mathrm{drt}}$–ID consists of the *undelayable atomic actions* $\{\underline{\underline{a}} \mid a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *time unit delay operator* σ_{rel}, the *“now” operator* ν_{rel}, the *(free) merge operator* $\|$, and the *left merge operator* $⫽$.

Definition 3.2.2 (Axioms of $\mathrm{PA}^-_{\mathrm{drt}}$–ID)
The process algebra $\mathrm{PA}^-_{\mathrm{drt}}$–ID is axiomatized by the axioms of $\mathrm{BPA}^-_{\mathrm{drt}}$–ID given in Definition 2.5.2 on page 22, Axioms DRTM1–DRTM4 shown in Table 24, and Axioms DRTM5–DRTM6 shown in Table 25 on the following page: $\mathrm{PA}^-_{\mathrm{drt}}$–ID = A1–A5 + DRT1–DRT5 + DCS1–DCS4 + DRTM1–DRTM6.

$x \parallel y = x ⫽ y + y ⫽ x$	DRTM1
$\underline{\underline{a}} ⫽ x = \underline{\underline{a}} \cdot x$	DRTM2
$\underline{\underline{a}} \cdot x ⫽ y = \underline{\underline{a}} \cdot (x \parallel y)$	DRTM3
$(x + y) ⫽ z = x ⫽ z + y ⫽ z$	DRTM4

Table 24: Axioms for the (free) merge.

Definition 3.2.3 (Semantics of $\mathrm{PA}^-_{\mathrm{drt}}$–ID)
The semantics of $\mathrm{PA}^-_{\mathrm{drt}}$–ID are given by the term deduction system $T(\mathrm{PA}^-_{\mathrm{drt}}\text{–ID})$ induced by the deduction rules for $\mathrm{BPA}^-_{\mathrm{drt}}$–ID given in Definition 2.5.4 on page 23 and the deduction rules for the (free) merge given in Table 26 on the following page.

$\sigma_{\mathrm{rel}}(x) \mathbin{\|\!\_} \nu_{\mathrm{rel}}(y) = \underline{\underline{\delta}}$	DRTM5
$\sigma_{\mathrm{rel}}(x) \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(y) + \sigma_{\mathrm{rel}}(z)) = \sigma_{\mathrm{rel}}(x \mathbin{\|\!\_} z)$	DRTM6

Table 25: Additional axioms for $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID.

$$\frac{x \xrightarrow{a} x'}{x \parallel y \xrightarrow{a} x' \parallel y} \qquad \frac{y \xrightarrow{a} y'}{x \parallel y \xrightarrow{a} x \parallel y'} \qquad \frac{x \xrightarrow{a} x'}{x \mathbin{\|\!\_} y \xrightarrow{a} x' \parallel y}$$

$$\frac{x \xrightarrow{a} \surd}{x \parallel y \xrightarrow{a} y} \qquad \frac{y \xrightarrow{a} \surd}{x \parallel y \xrightarrow{a} x} \qquad \frac{x \xrightarrow{a} \surd}{x \mathbin{\|\!\_} y \xrightarrow{a} y}$$

$$\frac{x \xrightarrow{\sigma} x',\ y \xrightarrow{\sigma} y'}{x \parallel y \xrightarrow{\sigma} x' \parallel y'} \qquad \frac{x \xrightarrow{\sigma} x',\ y \xrightarrow{\sigma} y'}{x \mathbin{\|\!\_} y \xrightarrow{\sigma} x' \mathbin{\|\!\_} y'}$$

Table 26: Deduction rules for the (free) merge.

Definition 3.2.4 (Bisimulation and Bisimulation Model for $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID)
Bisimulation for $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID and the corresponding bisimulation model are defined in the same way as for $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ and BPA respectively. Replace “$\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$” by “$\mathrm{PA}^{-}_{\mathrm{drt}}$–ID” in Definition 2.4.5 on page 14 and “BPA” by “$\mathrm{PA}^{-}_{\mathrm{drt}}$–ID” in Definition 2.2.11 on page 8.

Definition 3.2.5 (Basic Terms of $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID)
If we speak of basic terms in the context of $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID, we mean $(\sigma, \underline{\underline{\delta}})$-basic terms as defined in Definition 2.5.6 on page 23.

Definition 3.2.6 (Number of Symbols of a $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID Term)
We define $n(x)$, the number of symbols of x, inductively as follows:

(i). For $a \in A_\delta$, we define $n(\underline{\underline{a}}) = 1$,

(ii). for closed $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID terms x and y, we define $n(x + y) = n(x \cdot y) = n(x \parallel y) = n(x \mathbin{\|\!\_} y) = n(x) + n(y) + 1$,

(iii). for a closed $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID term x, we define $n(\sigma_{\mathrm{rel}}(x)) = n(\nu_{\mathrm{rel}}(x)) = n(x) + 1$.

Remark 3.2.7 (Proving Elimination using the Direct Method)
In some settings the term rewriting analysis method for proving elimination mentioned in Remark 2.4.10 on page 15 does not work, as the term rewriting system we arrive at is not *strongly* terminating. In these cases, we apply a direct method: we simply prove that for all closed terms elimination can be achieved by examining all possible cases. Although conceptually simple, this method often gives rise to very lengthy proofs, exponentially so for theories that contain many features. See Theorem 3.2.8 on the next page for an example of this method.

Theorem 3.2.8 (Elimination for $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID)
Let t be a closed PA^{-}_{drt}-ID term. Then there is a closed BPA^{-}_{drt}-ID term s such that PA^{-}_{drt}-ID $\vdash$ $s = t$.

Proof Let t be a closed $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID term. The theorem is proven by induction on $n(t)$ and case distinction on the general structure of t.

(i). $t \equiv \underline{\underline{a}}$ for some $a \in A_\delta$. Then t is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

(ii). $t \equiv t_1 + t_2$ for closed $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID terms t_1 and t_2. By induction there are closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID terms s_1 and s_2 such that $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_1 = s_1$ and $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_2 = s_2$. But then also $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_1 + t_2 = s_1 + s_2$ and $s_1 + s_2$ is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

(iii). $t \equiv t_1 \cdot t_2$ for closed $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID terms t_1 and t_2. This case is treated analogously to case (ii).

(iv). $t \equiv \sigma_{\mathrm{rel}}(t_1)$ for a closed $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID term t_1. This case is treated analogously to case (ii).

(v). $t \equiv \nu_{\mathrm{rel}}(t_1)$ for a closed $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID term t_1. This case is treated analogously to case (ii).

(vi). $t \equiv t_1 \mathbin{\lfloor\!\lfloor} t_2$ for closed $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID terms t_1 and t_2. By induction there are closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID terms s_1 and s_2 such that $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_1 = s_1$ and $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_2 = s_2$. By Theorem 2.5.12, the elimination theorem for $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID, there are basic terms r_1 and r_2 such that $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID $\vdash s_1 = r_1$ and $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID $\vdash s_2 = r_2$. But then also, $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_1 = r_1$, $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_2 = r_2$, and $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_1 \mathbin{\lfloor\!\lfloor} t_2 = r_1 \mathbin{\lfloor\!\lfloor} r_2$. We prove this case by induction on the structure of basic term r_1:

(a) $r_1 \equiv \underline{\underline{a}}$ for some $a \in A_\delta$. Then $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_1 \mathbin{\lfloor\!\lfloor} t_2 = r_1 \mathbin{\lfloor\!\lfloor} r_2 = \underline{\underline{a}} \mathbin{\lfloor\!\lfloor} r_2 = \underline{\underline{a}} \cdot r_2$, and $\underline{\underline{a}} \cdot r_2$ is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

(b) $r_1 \equiv \underline{\underline{a}} \cdot r_1'$ for some $a \in A_\delta$ and basic term r_1'. Then $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_1 \mathbin{\lfloor\!\lfloor} t_2 = r_1 \mathbin{\lfloor\!\lfloor} r_2 = \underline{\underline{a}} \cdot r_1' \mathbin{\lfloor\!\lfloor} r_2 = \underline{\underline{a}} \cdot (r_1' \parallel r_2)$. By the induction hypothesis there exists a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term p such that $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash r_1' \parallel r_2 = p$. Then, $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash$ $t_1 \mathbin{\lfloor\!\lfloor} t_2 = \underline{\underline{a}} \cdot (r_1' \parallel r_2) = \underline{\underline{a}} \cdot p$, and $\underline{\underline{a}} \cdot p$ is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

(c) $r_1 \equiv r_1' + r_1''$ for basic terms r_1' and r_1''. Then $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_1 \mathbin{\lfloor\!\lfloor} t_2 = r_1 \mathbin{\lfloor\!\lfloor} r_2 = (r_1' + r_1'') \mathbin{\lfloor\!\lfloor} r_2 = r_1' \mathbin{\lfloor\!\lfloor} r_2 + r_1'' \mathbin{\lfloor\!\lfloor} r_2$. By induction there exist closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID terms p_1 and p_2 such that $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash r_1' \mathbin{\lfloor\!\lfloor} r_2 = p_1$ and $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash r_1'' \mathbin{\lfloor\!\lfloor} r_2 = p_2$. Then also $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_1 \mathbin{\lfloor\!\lfloor} t_2 = r_1 \mathbin{\lfloor\!\lfloor} r_2 = r_1' \mathbin{\lfloor\!\lfloor} r_2 + r_1'' \mathbin{\lfloor\!\lfloor} r_2 = p_1 + p_2$, and $p_1 + p_2$ is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

(d) $r_1 \equiv \sigma_{\mathrm{rel}}(r_1')$ for a basic term r_1'. By Lemma 2.5.10 there is a basic term r_2' such that either $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash r_2 = \nu_{\mathrm{rel}}(r_2)$ or $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash r_2 = \nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')$ with $n(r_2') < n(r_2)$. With case analysis we obtain:

i. $r_2 = \nu_{\mathrm{rel}}(r_2)$. Then $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_1 \mathbin{\lfloor\!\lfloor} t_2 = r_1 \mathbin{\lfloor\!\lfloor} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{\lfloor\!\lfloor} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{\lfloor\!\lfloor} \nu_{\mathrm{rel}}(r_2) = \underline{\underline{\delta}}$, and $\underline{\underline{\delta}}$ is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

ii. $r_2 = \nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')$ for a basic term r_2'. Then $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_1 \mathbin{\lfloor\!\lfloor} t_2 = r_1 \mathbin{\lfloor\!\lfloor} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{\lfloor\!\lfloor} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{\lfloor\!\lfloor} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) = \sigma_{\mathrm{rel}}(r_1' \mathbin{\lfloor\!\lfloor} r_2')$. By the induction hypothesis there is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term p such that $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash r_1' \mathbin{\lfloor\!\lfloor} r_2' = p$. But then also $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID $\vdash t_1 \mathbin{\lfloor\!\lfloor} t_2 = \sigma_{\mathrm{rel}}(r_1' \mathbin{\lfloor\!\lfloor} r_2') = \sigma_{\mathrm{rel}}(p)$, and $\sigma_{\mathrm{rel}}(p)$ is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

(vii). $t \equiv t_1 \parallel t_2$ for closed $\mathrm{PA}^-_{\mathrm{drt}}$–ID terms t_1 and t_2. Then $\mathrm{PA}^-_{\mathrm{drt}}$–ID $\vdash t_1 \parallel t_2 = t_1 \mathbin{\lfloor\!\lfloor} t_2 + t_2 \mathbin{\lfloor\!\lfloor} t_1$. By (vi) there are closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID terms p_1 and p_2 such that $\mathrm{PA}^-_{\mathrm{drt}}$–ID $\vdash t_1 \mathbin{\lfloor\!\lfloor} t_2 = p_1$ and $\mathrm{PA}^-_{\mathrm{drt}}$–ID $\vdash t_2 \mathbin{\lfloor\!\lfloor} t_1 = p_2$. But then also $\mathrm{PA}^-_{\mathrm{drt}}$–ID $\vdash t_1 \parallel t_2 = t_1 \mathbin{\lfloor\!\lfloor} t_2 + t_2 \mathbin{\lfloor\!\lfloor} t_1 = p_1 + p_2$, and $p_1 + p_2$ is a closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID term.

■

Corollary 3.2.9 (Elimination for $\mathrm{PA}^-_{\mathrm{drt}}$–ID)
Let t be a closed PA^-_{drt}–ID term. Then there is a basic term s such that PA^-_{drt}–ID $\vdash s = t$.

Proof This follows immediately from:

(i). The elimination theorem for $\mathrm{PA}^-_{\mathrm{drt}}$–ID (see Theorem 3.2.8),

(ii). the elimination theorem for $\mathrm{BPA}^-_{\mathrm{drt}}$–ID (see Theorem 2.5.12),

(iii). the fact that all axioms of $\mathrm{BPA}^-_{\mathrm{drt}}$–ID are also contained in $\mathrm{PA}^-_{\mathrm{drt}}$–ID.

■

Remark 3.2.10 (Elimination for $\mathrm{PA}^-_{\mathrm{drt}}$–ID)
Elimination for $\mathrm{PA}^-_{\mathrm{drt}}$–ID is also claimed (without proof) in Theorem 3.2 of [11].

Theorem 3.2.11 (Soundness of $\mathrm{PA}^-_{\mathrm{drt}}$–ID)
The set of closed PA^-_{drt}–ID terms modulo bisimulation equivalence is a model of PA^-_{drt}–ID.

Proof Soundness is proven following the same lines as in the previous soundness theorems.

Axiom DRTM1 Take the relation:

$$R = \{(s,s), (s \parallel t, t \parallel s), (s \parallel t, s \mathbin{\lfloor\!\lfloor} t + t \mathbin{\lfloor\!\lfloor} s) \mid s,t \in C(\mathrm{PA}^-_{\mathrm{drt}}\text{–ID})\}$$

First we look at the transitions of the left-hand side:

(i). Suppose that $s \parallel t \xrightarrow{a} p$. First we look at the $(s \parallel t, t \parallel s)$ pairs. By inspection of the deduction rules we can conclude that either $s \xrightarrow{a} p_1$ and $p \equiv p_1 \parallel t$, or $t \xrightarrow{a} p_2$ and $p \equiv s \parallel p_2$, or $s \xrightarrow{a} \surd$ and $p \equiv t$, or $t \xrightarrow{a} \surd$ and $p \equiv s$. Therefore, either $t \parallel s \xrightarrow{a} t \parallel p_1$, or $t \parallel s \xrightarrow{a} p_2 \parallel s$, or $t \parallel s \xrightarrow{a} t$, or $t \parallel s \xrightarrow{a} s$ respectively, and note that $(p_1 \parallel t, t \parallel p_1) \in R$, $(s \parallel p_2, p_2 \parallel s) \in R$, $(t,t) \in R$, and $(s,s) \in R$.
Continuing with the $(s \parallel t, s \mathbin{\lfloor\!\lfloor} t + t \mathbin{\lfloor\!\lfloor} s)$ pairs, we also have either $s \mathbin{\lfloor\!\lfloor} t \xrightarrow{a} p_1 \parallel t$, or $t \mathbin{\lfloor\!\lfloor} s \xrightarrow{a} p_2 \parallel s$, or $s \mathbin{\lfloor\!\lfloor} t \xrightarrow{a} t$, or $t \mathbin{\lfloor\!\lfloor} s \xrightarrow{a} s$. Therefore, either $s \mathbin{\lfloor\!\lfloor} t + t \mathbin{\lfloor\!\lfloor} s \xrightarrow{a} p_1 \parallel t$, or $s \mathbin{\lfloor\!\lfloor} t + t \mathbin{\lfloor\!\lfloor} s \xrightarrow{a} p_2 \parallel s$, or $s \mathbin{\lfloor\!\lfloor} t + t \mathbin{\lfloor\!\lfloor} s \xrightarrow{a} t$, or $s \mathbin{\lfloor\!\lfloor} t + t \mathbin{\lfloor\!\lfloor} s \xrightarrow{a} s$ respectively, and again note that $(p_1 \parallel t, p_1 \parallel t) \in R$, $(s \parallel p_2, p_2 \parallel s) \in R$, $(t,t) \in R$, and $(s,s) \in R$.

(ii). Suppose that $s \parallel t \xrightarrow{a} \surd$. This case cannot occur.

(iii). Suppose that $s \parallel t \xrightarrow{\sigma} p$. First we look at the $(s \parallel t, t \parallel s)$ pairs. By inspection of the deduction rules we can conclude that $s \xrightarrow{\sigma} p_1$, $t \xrightarrow{\sigma} p_2$, and $p \equiv p_1 \parallel p_2$. Therefore, $t \parallel s \xrightarrow{\sigma} p_2 \parallel p_1$, and note that $(p_1 \parallel p_2, p_2 \parallel p_1) \in R$.
Continuing with the $(s \parallel t, s \mathbin{\lfloor\!\lfloor} t + t \mathbin{\lfloor\!\lfloor} s)$ pairs, we also have $s \mathbin{\lfloor\!\lfloor} t \xrightarrow{\sigma} p_1 \mathbin{\lfloor\!\lfloor} p_2$ and $t \mathbin{\lfloor\!\lfloor} s \xrightarrow{\sigma} p_2 \mathbin{\lfloor\!\lfloor} p_1$. Therefore, $s \mathbin{\lfloor\!\lfloor} t + t \mathbin{\lfloor\!\lfloor} s \xrightarrow{\sigma} p_1 \mathbin{\lfloor\!\lfloor} p_2 + p_2 \mathbin{\lfloor\!\lfloor} p_1$, and note that $(p_1 \parallel p_2, p_1 \mathbin{\lfloor\!\lfloor} p_2 + p_2 \mathbin{\lfloor\!\lfloor} p_1) \in R$.

Secondly, we look at the transitions of the right-hand side:

(i). Suppose that $t \parallel s \xrightarrow{a} p$. This case is handled in the same way as the corresponding (sub)case for the left-hand side shown above.

(ii). Suppose that $t \parallel s \xrightarrow{a} \surd$. This case cannot occur.

(iii). Suppose that $t \parallel s \xrightarrow{\sigma} p$. This case is handled in the same way as the corresponding (sub)case for the left-hand side shown above.

(iv). Suppose that $s \mathbin{⫩} t + t \mathbin{⫩} s \xrightarrow{a} p$. By inspection of the deduction rules we can conclude that either $s \xrightarrow{a} p_1$ and $p \equiv p_1 \parallel t$, or $t \xrightarrow{a} p_2$ and $p \equiv p_2 \parallel s$, or $s \xrightarrow{a} \surd$ and $p \equiv t$, or $t \xrightarrow{a} \surd$ and $p \equiv s$. Therefore, either $s \parallel t \xrightarrow{a} p_1 \parallel t$, or $s \parallel t \xrightarrow{a} s \parallel p_2$, or $s \parallel t \xrightarrow{a} t$, or $s \parallel t \xrightarrow{a} s$ respectively, and note that $(p_1 \parallel t, p_1 \parallel t) \in R$, $(s \parallel p_2, p_2 \parallel s) \in R$, $(t,t) \in R$, and $(s,s) \in R$.

(v). Suppose that $s \mathbin{⫩} t + t \mathbin{⫩} s \xrightarrow{a} p$. This case cannot occur.

(vi). Suppose that $s \mathbin{⫩} t + t \mathbin{⫩} s \xrightarrow{\sigma} p$. By inspection of the deduction rules we can conclude that $s \xrightarrow{\sigma} p_1$, $t \xrightarrow{\sigma} p_2$, and $p \equiv p_1 \mathbin{⫩} p_2 + p_2 \mathbin{⫩} p_1$. Since both s and t can perform a σ transition, we obtain $s \parallel t \xrightarrow{\sigma} p_1 \parallel p_2$, and note that $(p_1 \parallel p_2, p_1 \mathbin{⫩} p_2 + p_2 \mathbin{⫩} p_1) \in R$.

Axiom DRTM2 Take the relation:

$$R = \{(s,s), (\underline{\underline{a}} \mathbin{⫩} s, \underline{\underline{a}} \cdot s) \mid s \in C(\mathrm{PA}^{-}_{\mathrm{drt}}\text{–ID})\}$$

We look at the transitions of both sides at the same time. The only possible transition of the left-hand side is $\underline{\underline{a}} \mathbin{⫩} s \xrightarrow{a} s$, the only possible transition of the right-hand side is $\underline{\underline{a}} \cdot s \xrightarrow{a} s$, and note that $(s,s) \in R$.

Axiom DRTM3 Take the relation:

$$R = \{(s,s), (\underline{\underline{a}} \cdot s \mathbin{⫩} t, \underline{\underline{a}} \cdot (s \parallel t)) \mid s \in C(\mathrm{PA}^{-}_{\mathrm{drt}}\text{–ID})\}$$

We look at the transitions of both sides at the same time. The only possible transition of the left-hand side is $\underline{\underline{a}} \cdot s \mathbin{⫩} t \xrightarrow{a} s \parallel t$, the only possible transition of the right-hand side is $\underline{\underline{a}} \cdot (s \parallel t) \xrightarrow{a} s \parallel t$, and note that $(s \parallel t, s \parallel t) \in R$.

Axiom DRTM4 Take the relation:

$$R = \{(s,s), ((s+t) \mathbin{⫩} u, s \mathbin{⫩} u + t \mathbin{⫩} u) \mid s,t,u \in C(\mathrm{PA}^{-}_{\mathrm{drt}}\text{–ID})\}$$

First we look at the transitions of the left-hand side:

(i). Suppose $(s+t) \mathbin{⫩} u \xrightarrow{a} p$. By inspection of the deduction rules we can conclude that either $s \xrightarrow{a} p_1$ and $p \equiv p_1 \parallel u$, or $t \xrightarrow{a} p_2$ and $p \equiv p_2 \parallel u$, or $s \xrightarrow{a} \surd$ and $p \equiv u$, or $t \xrightarrow{a} \surd$ and $p \equiv u$. So, either $s \mathbin{⫩} u \xrightarrow{a} p_1 \parallel u$, or $t \mathbin{⫩} u \xrightarrow{a} p_2 \parallel u$, or $s \mathbin{⫩} u \xrightarrow{a} u$, or $t \mathbin{⫩} u \xrightarrow{a} u$. respectively. Therefore, either $s \mathbin{⫩} u + t \mathbin{⫩} u \xrightarrow{a} p_1 \parallel u$, or $s \mathbin{⫩} u + t \mathbin{⫩} u \xrightarrow{a} p_2 \parallel u$, or $s \mathbin{⫩} u + t \mathbin{⫩} u \xrightarrow{a} u$, or $s \mathbin{⫩} u + t \mathbin{⫩} u \xrightarrow{a} u$ respectively, and note that $(p_1 \parallel u, p_1 \parallel u) \in R$, $(p_2 \parallel u, p_2 \parallel u) \in R$, $(u,u) \in R$, and $(u,u) \in R$.

(ii). Suppose $(s+t) \mathbin{⫩} u \xrightarrow{a} \surd$. This case cannot occur.

(iii). Suppose $(s+t) \mathbin{⫫} u \xrightarrow{\sigma} p$. Then $s+t \xrightarrow{\sigma} p_1$, $u \xrightarrow{\sigma} p_2$, and $p \equiv p_1 \mathbin{⫫} p_2$. Then one of the following situations has occurred:

(a) $s \xrightarrow{\sigma} p_1$ and $t \not\xrightarrow{\sigma}$: then $s \mathbin{⫫} u \xrightarrow{\sigma} p_1 \mathbin{⫫} p_2$ and $t \mathbin{⫫} u \not\xrightarrow{\sigma}$. Therefore, $s \mathbin{⫫} u + t \mathbin{⫫} u \xrightarrow{\sigma} p_1 \mathbin{⫫} p_2$, and note that $(p_1 \mathbin{⫫} p_2, p_1 \mathbin{⫫} p_2) \in R$.

(b) $s \not\xrightarrow{\sigma}$ and $t \xrightarrow{\sigma} p_1$: this case is handled in the same way as the previous one.

(c) $s \xrightarrow{\sigma} q_1$, $t \xrightarrow{\sigma} q_2$, and $p_1 \equiv q_1 + q_2$: then $s \mathbin{⫫} u \xrightarrow{\sigma} q_1 \mathbin{⫫} p_2$ and $t \mathbin{⫫} u \xrightarrow{\sigma} q_2 \mathbin{⫫} p_2$. Therefore $s \mathbin{⫫} u + t \mathbin{⫫} u \xrightarrow{\sigma} q_1 \mathbin{⫫} p_2 + q_2 \mathbin{⫫} p_2$, and note that $((q_1+q_2) \mathbin{⫫} p_2, q_1 \mathbin{⫫} p_2 + q_2 \mathbin{⫫} p_2) \in R$.

Secondly, we look at the transitions of the right-hand side:

(i). Suppose $s \mathbin{⫫} u + t \mathbin{⫫} u \xrightarrow{a} p$. By inspection of the deduction rules we can conclude that either $s \xrightarrow{a} p_1$ and $p \equiv p_1 \parallel u$, or $t \xrightarrow{a} p_2$ and $p \equiv p_2 \parallel u$, or $s \xrightarrow{a} \surd$ and $p \equiv u$, or $t \xrightarrow{a} \surd$ and $p \equiv u$. Therefore, either $(s+t) \mathbin{⫫} u \xrightarrow{a} p_1 \parallel u$, or $(s+t) \mathbin{⫫} u \xrightarrow{a} p_2 \parallel u$, or $(s+t) \mathbin{⫫} u \xrightarrow{a} u$, or $(s+t) \mathbin{⫫} u \xrightarrow{a} u$, and note that $(p_1 \parallel u, p_1 \parallel u) \in R$, $(p_2 \parallel u, p_2 \parallel u) \in R$, $(u,u) \in R$, and $(u,u) \in R$.

(ii). Suppose $s \mathbin{⫫} u + t \mathbin{⫫} u \xrightarrow{a} \surd$. This case cannot occur.

(iii). Suppose $s \mathbin{⫫} u + t \mathbin{⫫} u \xrightarrow{\sigma} p$. Then this must be due to one of the following:

(a) $s \mathbin{⫫} u \xrightarrow{\sigma} p$ and $t \mathbin{⫫} u \not\xrightarrow{\sigma}$: then $s \xrightarrow{\sigma} q_1$, $u \xrightarrow{\sigma} q_2$, and $p \equiv q_1 \mathbin{⫫} q_2$. Therefore, $s+t \xrightarrow{\sigma} q_1$ and $(s+t) \mathbin{⫫} u \xrightarrow{\sigma} q_1 \mathbin{⫫} q_2$, and note that $(q_1 \mathbin{⫫} q_2, q_1 \mathbin{⫫} q_2) \in R$.

(b) $s \mathbin{⫫} u \not\xrightarrow{\sigma}$ and $t \mathbin{⫫} u \xrightarrow{\sigma} p$: this case is handled in the same way as the previous one.

(c) $s \mathbin{⫫} u \xrightarrow{\sigma} p_1$, $t \mathbin{⫫} u \xrightarrow{\sigma} p_2$, and $p \equiv p_1 + p_2$: then $s \xrightarrow{\sigma} q_1$, $t \xrightarrow{\sigma} q_2$, $u \xrightarrow{\sigma} q_3$, $p_1 \equiv q_1 \mathbin{⫫} q_3$, and $p_2 \equiv q_2 \mathbin{⫫} q_3$. Therefore, $s+t \xrightarrow{\sigma} q_1+q_2$ and $(s+t) \mathbin{⫫} u \xrightarrow{\sigma} (q_1+q_2) \mathbin{⫫} q_3$, and note that $((q_1+q_2) \mathbin{⫫} q_3, q_1 \mathbin{⫫} q_3 + q_2 \mathbin{⫫} q_3) \in R$.

Axiom DRTM5 Take the relation:

$$R = \{(\sigma_{\mathrm{rel}}(s) \mathbin{⫫} \nu_{\mathrm{rel}}(t), \underline{\underline{\delta}}) \mid s,t \in C(\mathrm{PA}^{-}_{\mathrm{drt}}\text{–ID})\}$$

We look at the transitions of both sides at the same time. Observe that there are no transitions possible on the left-hand side: $\sigma_{\mathrm{rel}}(s) \mathbin{⫫} \nu_{\mathrm{rel}}(t) \not\rightarrow$. Also for the right-hand side there are no transitions possible: $\underline{\underline{\delta}} \not\rightarrow$.

Axiom DRTM6 Take the relation:

$$R = \{(s,s), (\sigma_{\mathrm{rel}}(s) \mathbin{⫫} (\nu_{\mathrm{rel}}(t) + \sigma_{\mathrm{rel}}(u)), \sigma_{\mathrm{rel}}(s \mathbin{⫫} u)) \mid s,t,u \in C(\mathrm{PA}^{-}_{\mathrm{drt}}\text{–ID})\}$$

We look at the transitions of both sides at the same time. Observe that the only possible transition of the left-hand side is $\sigma_{\mathrm{rel}}(s) \mathbin{⫫} (\nu_{\mathrm{rel}}(t) + \sigma_{\mathrm{rel}}(u)) \xrightarrow{\sigma} s \mathbin{⫫} u$, and that the only possible transition of the right-hand side is $\sigma_{\mathrm{rel}}(s \mathbin{⫫} u) \xrightarrow{\sigma} s \mathbin{⫫} u$, and note that $(s \mathbin{⫫} u, s \mathbin{⫫} u) \in R$.

■

Remark 3.2.12 (Soundness of $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID)
Soundness of $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID is also claimed (without proof) in Theorem 3.3 of [11].

Remark 3.2.13 (Conservativity etc.)
In Theorem 3.2.15 and 3.2.16 below we use the concepts *conservative extension*, *operationally conservative extension*, *path format*, and *sum of term deduction systems*. For a formal definition of these concepts see Section 2.4.1 of [13], and the references given there.

Remark 3.2.14 (Proving Completeness using Verhoef's Theorem)
Instead of using the direct proof method (outlined in Remark 2.2.18 on page 9) to prove completeness, we can also use Verhoef's General Completeness Theorem [25] to derive completeness of a process theory P, given the completeness of a subtheory P' of P. In order to do this, we need to prove that P has the elimination property for P', and that P is a conservative extension of P'.

See Theorems 3.2.15 and 3.2.16 below for a example of this proof method. Furthermore, this method is used in the proofs of Theorems 3.4.15, 3.6.14, and 3.7.17.

Theorem 3.2.15 (Conservativity of $\mathrm{PA}^-_{\mathrm{drt}}$–ID with respect to $\mathrm{BPA}^-_{\mathrm{drt}}$–ID)
The equational specification PA^-_{drt}–ID is a conservative extension of the equational specification BPA^-_{drt}–ID.

Proof In order to prove conservativity it is sufficient to verify that the following conditions are satisfied:

(i). Bisimulation equivalence is definable in terms of predicate and relation symbols only,

(ii). $\mathrm{BPA}^-_{\mathrm{drt}}$–ID is a complete axiomatization with respect to the bisimulation equivalence model induced by $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID})$ (see Theorem 2.5.17),

(iii). $\mathrm{PA}^-_{\mathrm{drt}}$–ID is a sound axiomatization with respect to the bisimulation equivalence model induced by $T(\mathrm{PA}^-_{\mathrm{drt}}\text{–ID})$ (see Theorem 3.2.11),

(iv). $T(\mathrm{PA}^-_{\mathrm{drt}}\text{–ID})$ is an operationally conservative extension of $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID})$.

And in order for $T(\mathrm{PA}^-_{\mathrm{drt}}\text{–ID})$ indeed to be an operationally conservative extension of $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID})$ we must verify the following conditions:

(i). $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID})$ is a pure, well-founded term deduction system in path format,

(ii). $T(\mathrm{PA}^-_{\mathrm{drt}}\text{–ID})$ is a term deduction system in path format,

(iii). $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID}) \oplus T(\mathrm{PA}^-_{\mathrm{drt}}\text{–ID})$ is defined.

That the above properties hold can be trivially checked from the relevant definitions. ∎

Theorem 3.2.16 (Completeness of $\mathrm{PA}^-_{\mathrm{drt}}$–ID)
The equational specification PA^-_{drt}–ID is a complete axiomatization of the set of closed PA^-_{drt}–ID terms modulo bisimulation equivalence.

Proof By Verhoef's General Completeness Theorem (see [25], or Theorem 2.4.26 of [13]) this follows immediately from:

(i). $\mathrm{PA}^-_{\mathrm{drt}}$–ID has the elimination property for $\mathrm{BPA}^-_{\mathrm{drt}}$–ID (see Theorem 3.2.8), (ii). $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID is a conservative extension of $\mathrm{BPA}^{-}_{\mathrm{drt}}$-ID (see Theorem 3.2.15).

■

Remark 3.2.17 (Completeness of $\mathbf{PA}^{-}_{\mathbf{drt}}$-ID)
Completeness of $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID is also claimed (without proof) in Theorem 3.3 of [11].

3.3 Soundness and Completeness of $\mathbf{PA}^{-}_{\mathbf{drt}}$-ID′

Definition 3.3.1 (Signature of $\mathbf{PA}^{-}_{\mathbf{drt}}$-ID′)
The signature of $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′ is identical to the signature of $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID as given in Definition 3.2.1; it consists of the *undelayable atomic actions* $\{\underline{\underline{a}}|a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *time unit delay operator* σ_{rel}, the *"now" operator* ν_{rel}, the *(free) merge operator* $\|$, and the *left merge operator* $\mathbin{|\!\!\lfloor}$.

Definition 3.3.2 (Axioms of $\mathbf{PA}^{-}_{\mathbf{drt}}$-ID′)
The process algebra $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′ is axiomatized by the axioms of $\mathrm{BPA}^{-}_{\mathrm{drt}}$-ID given in Definition 2.5.2 on page 22, Axioms DRTM1–DRTM4 shown in Table 24 on page 69, and Axioms DRTM7–DRTM11 shown in Table 27: $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′ = A1–A5 + DRT1–DRT5 + DCS1–DCS4 + DRTM1–DRTM4 + DRTM7–DRTM11.

$\sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} \underline{\underline{a}} = \underline{\underline{\delta}}$	DRTM7
$\sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} \underline{\underline{a}} \cdot y = \underline{\underline{\delta}}$	DRTM8
$\sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} (\underline{\underline{a}} + y) = \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} y$	DRTM9
$\sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} (\underline{\underline{a}} \cdot y + z) = \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} z$	DRTM10
$\sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} \sigma_{\mathrm{rel}}(y) = \sigma_{\mathrm{rel}}(x \mathbin{\|\!\!\lfloor} y)$	DRTM11

Table 27: Additional axioms for $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′.

Definition 3.3.3 (Semantics of $\mathbf{PA}^{-}_{\mathbf{drt}}$-ID′)
The semantics of $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′ are given by the term deduction system $T(\mathrm{PA}^{-}_{\mathrm{drt}}\text{-ID}')$ which is identical to the term deduction system $T(\mathrm{PA}^{-}_{\mathrm{drt}}\text{-ID})$ given in Definition 3.2.3 on page 69.

Definition 3.3.4 (Bisimulation and Bisimulation Model for $\mathbf{PA}^{-}_{\mathbf{drt}}$-ID′)
Bisimulation for $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′ and the corresponding bisimulation model are defined in the same way as for $\mathrm{BPA}^{-}_{\mathrm{drt}}$-$\delta$ and BPA respectively. Replace "$\mathrm{BPA}^{-}_{\mathrm{drt}}$-$\delta$" by "$\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′" in Definition 2.4.5 on page 14 and "BPA" by "$\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′" in Definition 2.2.11 on page 8.

Definition 3.3.5 (Basic Terms of $\mathbf{PA}^{-}_{\mathbf{drt}}$-ID′)
If we speak of basic terms in the context of $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′, we mean $(\sigma, \underline{\underline{\delta}})$-basic terms as defined in Definition 2.5.6 on page 23.

Definition 3.3.6 (Number of Symbols of a $\mathbf{PA}^{-}_{\mathbf{drt}}$-ID′ Term)
We define $n(x)$, the number of symbols of x, inductively as follows:

(i). For $a \in A_\delta$, we define $n(\underline{\underline{a}}) = 1$,

(ii). for closed $\mathrm{PA}^-_{\mathrm{drt}}$–ID′ terms x and y, we define $n(x + y) = n(x \cdot y) = n(x \parallel y) = n(x \mathbin{\lfloor\!\lfloor} y) = n(x) + n(y) + 1$,

(iii). for a closed $\mathrm{PA}^-_{\mathrm{drt}}$–ID′ term x, we define $n(\sigma_{\mathrm{rel}}(x)) = n(\nu_{\mathrm{rel}}(x)) = n(x) + 1$.

Theorem 3.3.7 (Elimination for $\mathbf{PA}^-_{\mathbf{drt}}$–ID′)
Let t be a closed PA^-_{drt}-ID′ term. Then there is a closed BPA^-_{drt}-ID term s such that PA^-_{drt}-ID′ $\vdash s = t$.

Proof First a term rewriting system is given which is proven to be strongly normalizing. Thereafter, it is shown that every normal form of a closed $\mathrm{PA}^-_{\mathrm{drt}}$–ID′ term is a closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID term.

The term rewriting system used is given in Table 28. Note that we have added natural number subscripts n to the merge operators, in order to deal with the mutually recursive nature of definition of these operators. For a description of this technique, and a rigorous formal justification of its use, see Theorem 3.2.3 of [13] and the references given there.

$x \parallel_n y \to x \mathbin{\lfloor\!\lfloor}_n y + y \mathbin{\lfloor\!\lfloor}_n x$	RDRTM1
$\underline{\underline{a}} \mathbin{\lfloor\!\lfloor}_n x \to \underline{\underline{a}} \cdot x$	RDRTM2
$\underline{\underline{a}} \cdot x \mathbin{\lfloor\!\lfloor}_{n+1} y \to \underline{\underline{a}} \cdot (x \parallel_n y)$	RDRTM3
$(x + y) \mathbin{\lfloor\!\lfloor}_n z \to x \mathbin{\lfloor\!\lfloor}_n z + y \mathbin{\lfloor\!\lfloor}_n z$	RDRTM4
$\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n \underline{\underline{a}} \to \underline{\underline{\delta}}$	RDRTM7
$\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n \underline{\underline{a}} \cdot y \to \underline{\underline{\delta}}$	RDRTM8
$\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n (\underline{\underline{a}} + y) \to \sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n y$	RDRTM9
$\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n (\underline{\underline{a}} \cdot y + z) \to \sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n z$	RDRTM10
$\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n \sigma_{\mathrm{rel}}(y) \to \sigma_{\mathrm{rel}}(x \mathbin{\lfloor\!\lfloor}_n y)$	RDRTM11

Table 28: Additional rewriting rules for $\mathrm{PA}^-_{\mathrm{drt}}$–ID′.

Using the method of the lexicographical path ordering we prove that the term rewriting system associated with $\mathrm{PA}^-_{\mathrm{drt}}$–ID′ is strongly normalizing. Thereto, the operator $\cdot$ is assigned the lexicographical status of the first argument, and the following well-founded ordering on constant and function symbols is used:

$$\underline{\underline{a}} < \sigma_{\mathrm{rel}} < + < \cdot < \mathbin{\lfloor\!\lfloor}_2 < \parallel_2 < \mathbin{\lfloor\!\lfloor}_3 < \cdots < \mathbin{\lfloor\!\lfloor}_n < \parallel_n < \mathbin{\lfloor\!\lfloor}_{n+1} < \ldots$$

That the left-hand side of every rewriting rule is bigger than the right-hand side with respect to the ordering $\succ_{\mathrm{lpo}}$, is shown by the following reductions:

$$\begin{aligned} x \parallel_n y &\succ_{\mathrm{lpo}} x \parallel_n^\star y \succ_{\mathrm{lpo}} x \parallel_n^\star y + x \parallel_n^\star y \\ &\succ_{\mathrm{lpo}} (x \parallel_n^\star y) \mathbin{\lfloor\!\lfloor}_n (x \parallel_n^\star y) + (x \parallel_n^\star y) \mathbin{\lfloor\!\lfloor}_n (x \parallel_n^\star y) \\ &\succ_{\mathrm{lpo}} x \mathbin{\lfloor\!\lfloor}_n y + y \mathbin{\lfloor\!\lfloor}_n x \end{aligned}$$

$$
\begin{aligned}
\underline{\underline{a}} \lfloor\!\lfloor_n x &\succ_{\mathrm{lpo}} \underline{\underline{a}} \lfloor\!\lfloor_n^\star x \succ_{\mathrm{lpo}} \underline{\underline{a}} \lfloor\!\lfloor_n^\star x \cdot \underline{\underline{a}} \lfloor\!\lfloor_n^\star x \\
&\succ_{\mathrm{lpo}} \underline{\underline{a}} \cdot x \\
\underline{\underline{a}} \cdot x \lfloor\!\lfloor_{n+1} y &\succ_{\mathrm{lpo}} \underline{\underline{a}} \cdot x \lfloor\!\lfloor_{n+1}^\star y \succ_{\mathrm{lpo}} (\underline{\underline{a}} \cdot x \lfloor\!\lfloor_{n+1}^\star y) \cdot (\underline{\underline{a}} \cdot x \lfloor\!\lfloor_{n+1}^\star y) \\
&\succ_{\mathrm{lpo}} (\underline{\underline{a}} \cdot x) \cdot ((\underline{\underline{a}} \cdot x \lfloor\!\lfloor_{n+1}^\star y) \parallel_n (\underline{\underline{a}} \cdot x \lfloor\!\lfloor_{n+1}^\star y)) \\
&\succ_{\mathrm{lpo}} (\underline{\underline{a}} \cdot x) \cdot (\underline{\underline{a}} \cdot x \parallel_n y) \succ_{\mathrm{lpo}} (\underline{\underline{a}} \cdot^\star x) \cdot (\underline{\underline{a}} \cdot^\star x \parallel_n y) \\
&\succ_{\mathrm{lpo}} \underline{\underline{a}} \cdot (x \parallel_n y) \\
(x+y) \lfloor\!\lfloor_n z &\succ_{\mathrm{lpo}} (x+y) \lfloor\!\lfloor_n^\star z \succ_{\mathrm{lpo}} (x+y) \lfloor\!\lfloor_n^\star z + (x+y) \lfloor\!\lfloor_n^\star z \\
&\succ_{\mathrm{lpo}} (x +^\star y) \lfloor\!\lfloor_n z + (x +^\star y) \lfloor\!\lfloor_n z \succ_{\mathrm{lpo}} (x \lfloor\!\lfloor_n z) + (y \lfloor\!\lfloor_n z) \\
\sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n \underline{\underline{a}} &\succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n^\star \underline{\underline{a}} \\
&\succ_{\mathrm{lpo}} \underline{\underline{\delta}} \\
\sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n \underline{\underline{a}} \cdot y &\succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n^\star \underline{\underline{a}} \cdot y \\
&\succ_{\mathrm{lpo}} \underline{\underline{\delta}} \\
\sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n (\underline{\underline{a}} + y) &\succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n^\star (\underline{\underline{a}} + y) \succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n (\underline{\underline{a}} +^\star y) \\
&\succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n y \\
\sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n (\underline{\underline{a}} \cdot y + z) &\succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n^\star (\underline{\underline{a}} \cdot y + z) \succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n (\underline{\underline{a}} \cdot y +^\star z) \\
&\succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n z \\
\sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n \sigma_{\mathrm{rel}}(y) &\succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n^\star \sigma_{\mathrm{rel}}(y) \succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x) \lfloor\!\lfloor_n^\star \sigma_{\mathrm{rel}}(y)) \\
&\succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(\sigma_{\mathrm{rel}}^\star(x) \lfloor\!\lfloor_n \sigma_{\mathrm{rel}}^\star(y)) \succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x \lfloor\!\lfloor_n y)
\end{aligned}
$$

It remains to prove that every normal form of a closed $\mathrm{PA}^-_{\mathrm{drt}}$–ID′ term is a closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID term. We prove this as follows: suppose that s is a normal form of a closed $\mathrm{PA}^-_{\mathrm{drt}}$–ID′ term, and furthermore suppose that s is not a closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID term. Now consider the smallest subterm s' of s that is not a closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID term. Then, s' must be of the form $s' \equiv s_1 \parallel s_2$ or of the form $s' \equiv s_1 \lfloor\!\lfloor s_2$ for closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID terms s_1 and s_2. By the elimination theorem for $\mathrm{BPA}^-_{\mathrm{drt}}$–ID, Theorem 2.6.12, we may assume that s_1 and s_2 are basic terms. Now in the first case, $s' \equiv s_1 \parallel s_2$, clearly rewriting rule RDRTM1 is applicable. This contradicts the assumption that s is a normal form, so this case cannot occur. In the second case, $s' \equiv s_1 \lfloor\!\lfloor s_2$, it is not clear at first sight that a contradiction can be derived. So, the following cases have to be considered for basic term s_1:

(i). $s_1 \equiv \underline{\underline{a}}$ for some $a \in A_\delta$. Then rewriting rule RDRTM2 is applicable.

(ii). $s_1 \equiv \underline{\underline{a}} \cdot s_1'$ for some $a \in A_\delta$ and closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID term s_1'. Clearly, rewriting rule RDRTM3 is applicable.

(iii). $s_1 \equiv s_1' + s_1''$ for some closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID terms s_1' and s_1''. This time rewriting rule RDRTM4 is applicable.

(iv). $s_1 \equiv \sigma_{\mathrm{rel}}(s_1')$ for some closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID term s_1'. The following cases can be considered for the general form of basic term s_2:

  (a) $s_2 \equiv \underline{\underline{a}}$ for some $a \in A_\delta$. In this case rewriting rule RDRTM7 is applicable.

  (b) $s_2 \equiv \underline{\underline{a}} \cdot s_2'$ for some $a \in A$ and basic term s_2'. In this case rewriting rule RDRTM8 is applicable.

(c) $s_2 \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot s_{2,i} + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \sigma_{\mathrm{rel}}(s'_{2,k})$ for $m, n, p \in \mathbb{N}$, $a_i, b_j \in A_\delta$, and $s_{2,i}$ and $s'_{2,k}$ basic terms. In this case at least one of the rewriting rules RDRTM9, RDRTM10, and RDRTM11 is applicable.

In every case a rewriting rule is applicable. Therefore s' is not a normal form.

We see that in both cases s' is not a normal form. But this contradicts the assumption that s is a normal form. From this contradiction we conclude that s does not contain a merge operator. Therefore s must be closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID term, which had to be proven. ■

Corollary 3.3.8 (Elimination for $\mathbf{PA}^-_{\mathbf{drt}}$–ID′)
Let t be a closed PA^-_{drt}-ID′ term. Then there is a basic term s such that PA^-_{drt}-ID′ $\vdash s = t$.

Proof This follows immediately from:

(i). The elimination theorem for $\mathrm{PA}^-_{\mathrm{drt}}$–ID′ (see Theorem 3.3.7),

(ii). the elimination theorem for $\mathrm{BPA}^-_{\mathrm{drt}}$–ID (see Theorem 2.5.12),

(iii). the fact that all axioms of $\mathrm{BPA}^-_{\mathrm{drt}}$–ID are also contained in $\mathrm{PA}^-_{\mathrm{drt}}$–ID′.

■

Remark 3.3.9 (Elimination for $\mathbf{PA}^-_{\mathbf{drt}}$–ID′)
Elimination for a slightly different version of $\mathrm{PA}^-_{\mathrm{drt}}$–ID′ is also claimed (without proof) in Theorem 3.4.2 of [13] (where $\mathrm{PA}^-_{\mathrm{drt}}$–ID′ is called $\mathrm{PA}_{\delta\mathrm{dt}}$).

Definition 3.3.10 (Ground Equivalence of Equational Specifications)
Two equational specifications $L_1 = (\Sigma_1, E_1)$ and $L_2 = (\Sigma_2, E_2)$ are called ground equivalent if they have identical signatures, i.e. $\Sigma_1 = \Sigma_2$, and the same equalities over closed terms hold in both systems, i.e. for all closed terms s and t over the signature Σ_1 (or Σ_2) we have $L_1 \vdash s = t$ iff $L_2 \vdash s = t$.

Definition 3.3.11 (Ground Equivalence of Term Deduction Systems)
Two equational specifications $T_1 = (\Sigma_1, D_1)$ and $T_2 = (\Sigma_2, D_2)$ are called ground equivalent if they have identical signatures, i.e. $\Sigma_1 = \Sigma_2$, and the same equalities over closed terms hold in both systems, i.e. for all closed terms s and t over the signature Σ_1 (or Σ_2) we have $s \sim_{T_1} t$ iff $s \sim_{T_2} t$.

Theorem 3.3.12 (Ground Equivalence of $\mathbf{PA}^-_{\mathbf{drt}}$–ID and $\mathbf{PA}^-_{\mathbf{drt}}$–ID′)
For all closed PA^-_{drt}-ID terms s and t we have PA^-_{drt}-ID $\vdash s = t$ if and only if PA^-_{drt}-ID′ $\vdash s = t$.

Proof It suffices to show that, for closed terms, every axiom of $\mathrm{PA}^-_{\mathrm{drt}}$–ID′ is derivable from the axioms of $\mathrm{PA}^-_{\mathrm{drt}}$–ID, and vice versa, that, for closed terms, every axiom of $\mathrm{PA}^-_{\mathrm{drt}}$–ID is derivable from the axioms of $\mathrm{PA}^-_{\mathrm{drt}}$–ID′. We can restrict ourselves to the axioms that are not shared by both theories.

Part I

First, we show that Axioms DRTM7–DRTM11 of $\mathrm{PA}^-_{\mathrm{drt}}$–ID′ are derivable in $\mathrm{PA}^-_{\mathrm{drt}}$–ID:

Axiom DRTM7

$$\mathrm{PA}^-_{\mathrm{drt}}\text{–ID} \vdash \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \underline{\underline{a}} = \underline{\underline{\delta}}$$

Consider the following derivation:

$$\mathrm{PA}^-_{\mathrm{drt}}\text{–ID} \vdash \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \underline{\underline{a}} = \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \nu_{\mathrm{rel}}(\underline{\underline{a}}) = \underline{\underline{\delta}}$$

Axiom DRTM8

$$\mathrm{PA}^-_{\mathrm{drt}}\text{–ID} \vdash \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \underline{\underline{a}} \cdot y = \underline{\underline{\delta}}$$

Consider the following derivation:

$$\mathrm{PA}^-_{\mathrm{drt}}\text{–ID} \vdash \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \underline{\underline{a}} \cdot y = \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot y = \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \nu_{\mathrm{rel}}(\underline{\underline{a}} \cdot y) = \underline{\underline{\delta}}$$

Axiom DRTM9

$$\mathrm{PA}^-_{\mathrm{drt}}\text{–ID} \vdash \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} (\underline{\underline{a}} + y) = \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} y$$

By the Lemma 2.5.10, there is a basic terms s such that either $\mathrm{PA}^-_{\mathrm{drt}}\text{–ID} \vdash y = \nu_{\mathrm{rel}}(y)$ or $\mathrm{PA}^-_{\mathrm{drt}}\text{–ID} \vdash y = \nu_{\mathrm{rel}}(y) + \sigma_{\mathrm{rel}}(s)$. So, there are two cases to be distinguished.

(i). $\mathrm{PA}^-_{\mathrm{drt}}\text{–ID} \vdash y = \nu_{\mathrm{rel}}(y)$. Then we have:

$$\begin{aligned}
\mathrm{PA}^-_{\mathrm{drt}}\text{–ID} \vdash\ & \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} (\underline{\underline{a}} + y) = \\
& \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} (\underline{\underline{a}} + \nu_{\mathrm{rel}}(y)) = \\
& \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \nu_{\mathrm{rel}}(y)) = \\
& \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \nu_{\mathrm{rel}}(\underline{\underline{a}} + y) = \\
& \underline{\underline{\delta}} = \\
& \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \nu_{\mathrm{rel}}(y) = \\
& \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} y
\end{aligned}$$

(ii). $\mathrm{PA}^-_{\mathrm{drt}}\text{–ID} \vdash y = \nu_{\mathrm{rel}}(y) + \sigma_{\mathrm{rel}}(s)$. Then we have:

$$\begin{aligned}
\mathrm{PA}^-_{\mathrm{drt}}\text{–ID} \vdash\ & \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} (\underline{\underline{a}} + y) = \\
& \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} (\underline{\underline{a}} + \nu_{\mathrm{rel}}(y) + \sigma_{\mathrm{rel}}(s)) = \\
& \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \nu_{\mathrm{rel}}(y) + \sigma_{\mathrm{rel}}(s)) = \\
& \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} (\nu_{\mathrm{rel}}(\underline{\underline{a}} + y) + \sigma_{\mathrm{rel}}(s)) = \\
& \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \sigma_{\mathrm{rel}}(s) = \\
& \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} (\nu_{\mathrm{rel}}(y) + \sigma_{\mathrm{rel}}(s)) = \\
& \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} y
\end{aligned}$$

Axiom DRTM10

$$\mathrm{PA}^-_{\mathrm{drt}}\text{–ID} \vdash \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} (\underline{\underline{a}} \cdot y + z) = \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} z$$

Handled in the same way as the previous case.

Axiom DRTM11

$$\mathrm{PA}^{-}_{\mathrm{drt}}\text{–ID} \vdash \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} \sigma_{\mathrm{rel}}(y) = \sigma_{\mathrm{rel}}(x \mathbin{\|\!\!\lfloor} y)$$

Consider the following derivation: $\mathrm{PA}^{-}_{\mathrm{drt}}\text{–ID} \vdash \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} \sigma_{\mathrm{rel}}(y) = \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} (\underline{\underline{\delta}} + \sigma_{\mathrm{rel}}(y)) = \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} (\nu_{\mathrm{rel}}(\underline{\underline{\delta}}) + \sigma_{\mathrm{rel}}(y)) = \sigma_{\mathrm{rel}}(x \mathbin{\|\!\!\lfloor} y)$.

Part II

Secondly, we show that Axioms DRTM5–DRTM6 of $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID are derivable in $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID′:

Axiom DRTM5

$$\mathrm{PA}^{-}_{\mathrm{drt}}\text{–ID}' \vdash \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} \nu_{\mathrm{rel}}(y) = \underline{\underline{\delta}}$$

Use the general form of basic term y. Take:

$$y \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot t_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \sigma_{\mathrm{rel}}(u_k)$$

for $m, n, p \in \mathbb{N}$, $a_i, b_j \in A_\delta$, and basic terms t_i and u_k. Then we have:

$$\begin{aligned}
&\mathrm{PA}^{-}_{\mathrm{drt}}\text{–ID}' \vdash \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} \nu_{\mathrm{rel}}(y) = \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} \nu_{\mathrm{rel}}\left(\sum_{i<m} \underline{\underline{a_i}} \cdot t_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \sigma_{\mathrm{rel}}(u_k)\right) = \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} \left(\sum_{i<m} \nu_{\mathrm{rel}}(\underline{\underline{a_i}} \cdot t_i) + \sum_{j<n} \nu_{\mathrm{rel}}(\underline{\underline{b_j}}) + \sum_{k<p} \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(u_k))\right) = \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} \left(\sum_{i<m} \nu_{\mathrm{rel}}(\underline{\underline{a_i}}) \cdot t_i + \sum_{j<n} \nu_{\mathrm{rel}}(\underline{\underline{b_j}}) + \sum_{k<p} \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(u_k))\right) = \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} \left(\sum_{i<m} \underline{\underline{a_i}} \cdot t_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \underline{\underline{\delta}}\right) = \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} \left(\sum_{i<m} \underline{\underline{a_i}} \cdot t_i + \sum_{j<n} \underline{\underline{b_j}} + \underline{\underline{\delta}}\right) = \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} \underline{\underline{\delta}} = \\
&\qquad \underline{\underline{\delta}}
\end{aligned}$$

Axiom DRTM6

$$\mathrm{PA}^{-}_{\mathrm{drt}}\text{–ID}' \vdash \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\!\lfloor} (\nu_{\mathrm{rel}}(y) + \sigma_{\mathrm{rel}}(z)) = \sigma_{\mathrm{rel}}(x \mathbin{\|\!\!\lfloor} z)$$

Use the general form of basic term y. Take:

$$y \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot t_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \sigma_{\mathrm{rel}}(u_k)$$

for $m, n, p \in \mathbb{N}$, $a_i, b_j \in A_\delta$, and basic terms t_i and u_k. Then we have:

$$
\begin{aligned}
&\mathrm{PA}^{-}_{\mathrm{drt}}\text{-ID}' \vdash \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} (\nu_{\mathrm{rel}}(y) + \sigma_{\mathrm{rel}}(z)) = \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \left(\nu_{\mathrm{rel}}\left(\sum_{i<m} \underline{\underline{a_i}} \cdot t_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \sigma_{\mathrm{rel}}(u_k)\right) + \sigma_{\mathrm{rel}}(z)\right) = \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \left(\sum_{i<m} \nu_{\mathrm{rel}}(\underline{\underline{a_i}} \cdot t_i) + \sum_{j<n} \nu_{\mathrm{rel}}(\underline{\underline{b_j}}) + \sum_{k<p} \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(u_k)) + \sigma_{\mathrm{rel}}(z)\right) = \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \left(\sum_{i<m} \nu_{\mathrm{rel}}(\underline{\underline{a_i}}) \cdot t_i + \sum_{j<n} \nu_{\mathrm{rel}}(\underline{\underline{b_j}}) + \sum_{k<p} \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(u_k)) + \sigma_{\mathrm{rel}}(z)\right) = \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \left(\sum_{i<m} \underline{\underline{a_i}} \cdot t_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \underline{\underline{\delta}} + \sigma_{\mathrm{rel}}(z)\right) = \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \left(\sum_{i<m} \underline{\underline{a_i}} \cdot t_i + \sum_{j<n} \underline{\underline{b_j}} + \sigma_{\mathrm{rel}}(z)\right) = \\
&\qquad \sigma_{\mathrm{rel}}(x) \mathbin{\|\!\lfloor} \sigma_{\mathrm{rel}}(z) = \\
&\qquad \sigma_{\mathrm{rel}}(x \mathbin{\|\!\lfloor} z)
\end{aligned}
$$

■

Theorem 3.3.13 (Ground Equivalence of $T(\mathbf{PA}^{-}_{\mathbf{drt}}\text{–}\mathbf{ID})$ and $T(\mathbf{PA}^{-}_{\mathbf{drt}}\text{–}\mathbf{ID}'))$
The term deduction systems $T(PA^{-}_{drt}\text{-}ID)$ and $T(PA^{-}_{drt}\text{-}ID')$ are ground equivalent.

Proof Both term deduction systems have the same signature and the same set of deduction rules. Then it is trivial that the same equalities hold between closed terms. ■

Remark 3.3.14 (Proving Soundness and Completeness using Ground Equivalence)
Next to the direct method for proving completeness (outlined in Remark 2.2.18 on page 9), the indirect method (outlined in Remark 2.6.20 on page 44, or Verhoef's method (outlined in Remark 3.2.14 on page 75), we can also derive completeness of a process theory P from the given completeness of a "sufficiently similar" process theory P'. As it turns out, the intuitive notion "sufficiently similar" can be formalized by the notion of ground equivalence. Using this method, we can also derive the soundness of P.

See Theorem 3.3.15 below for technical details, and the proofs of Corollaries 3.3.17, 3.3.19, 3.5.12, and 3.5.14 for applications of this proof method.

Theorem 3.3.15 (Soundness and Completeness)
Let L_1 and L_2 be ground equivalent equational specifications. Let T_1 and T_2 be ground equivalent term deduction systems. Then, L_1 is a sound axiomatization with respect to the bisimulation equivalence model induced by T_1 if and only if L_2 is a sound axiomatization with respect to the bisimulation equivalence model induced by T_2. Also, L_1 is a complete axiomatization with respect to the bisimulation equivalence model induced by T_1 if and only if L_2 is a complete axiomatization with respect to the bisimulation equivalence model induced by T_2.

Proof Let L_1 and L_2 be ground equivalent equational specifications. Let T_1 and T_2 be ground equivalent term deduction systems. Suppose that L_1 is a sound axiomatization with respect to the bisimulation equivalence model induced by T_1. Let s and t be closed terms. Suppose that $L_2 \vdash s = t$. By the ground equivalence of L_1 and L_2 we obtain $L_1 \vdash s = t$. Using that L_1 is a sound axiomatization with respect to the bisimulation equivalence model induced by T_1 we have that $s \sim_{T_1} t$. By the ground equivalence of T_1 and T_2 it follows that $s \sim_{T_2} t$. The proof in the other direction is analogous. The second part of the theorem is proven analogously. ∎

Remark 3.3.16 (Soundness and Completeness versus Elimination)
Note that it is useless to extend Theorem 3.3.15 to include an elimination result, as in general elimination is used to prove ground equivalence between L_1 and L_2. In the proof of Theorem 3.3.12 on page 79, for example, we use elimination to allow ourselves to restrict the proof to basic terms.

Corollary 3.3.17 (Soundness of $\mathbf{PA}^{-}_{\mathbf{drt}}$-ID′)
The set of closed PA^{-}_{drt}-ID′ terms modulo bisimulation equivalence is a model of PA^{-}_{drt}-ID′.

Proof This follows immediately from the following observations and Theorem 3.3.15:

(i). $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID and $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′ are ground equivalent equational specifications (see Theorem 3.3.12),

(ii). $T(\mathrm{PA}^{-}_{\mathrm{drt}}\text{-ID})$ and $T(\mathrm{PA}^{-}_{\mathrm{drt}}\text{-ID}')$ are ground equivalent term deduction systems (see Theorem 3.3.13),

(iii). Soundness of $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID (see Theorem 3.2.11).

∎

Remark 3.3.18 (Soundness of $\mathbf{PA}^{-}_{\mathbf{drt}}$-ID′)
Soundness of a slightly different version of $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′ is also claimed (without proof) in Theorem 3.4.3 of [13] (where $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′ is called $\mathrm{PA}_{\delta\mathrm{dt}}$).

Corollary 3.3.19 (Completeness of $\mathbf{PA}^{-}_{\mathbf{drt}}$-ID′)
The equational specification PA^{-}_{drt}-ID′ is a complete axiomatization of the set of closed PA^{-}_{drt}-ID′ terms modulo bisimulation equivalence.

Proof This follows immediately from the following observations and Theorem 3.3.15:

(i). $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID and $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′ are ground equivalent equational specifications (see Theorem 3.3.12),

(ii). $T(\mathrm{PA}^{-}_{\mathrm{drt}}\text{-ID})$ and $T(\mathrm{PA}^{-}_{\mathrm{drt}}\text{-ID}')$ are ground equivalent term deduction systems (see Theorem 3.3.13),

(iii). Completeness of $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID (see Theorem 3.2.16).

∎

Remark 3.3.20 (Completeness of $\mathbf{PA}^{-}_{\mathbf{drt}}$-ID′)
Completeness of a slightly different version of $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′ is also claimed (without proof) in Theorem 3.4.5 of [13] (where $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID′ is called $\mathrm{PA}_{\delta\mathrm{dt}}$).

3.4 Soundness and Completeness of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID

Definition 3.4.1 (Communication Function)
For this section, and all sections to come, we presume the existence of a fixed, commutative, associative, complete function $\gamma : A_\delta \times A_\delta \to A_\delta$, that can be considered a parameter of the respective theories. The function γ has to be *strict* in the sense that it should always evaluate to δ when one or both of its parameters is δ.

Definition 3.4.2 (Signature of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID)
The signature of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID consists of the *undelayable atomic actions* $\{\underline{\underline{a}} | a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *time unit delay operator* σ_{rel}, the *"now" operator* ν_{rel}, the *(communicating) merge operator* $\|$, the *left merge operator* $\mathbin{\|\!\_}$, and the *communication merge operator* $\mid$.

Definition 3.4.3 (Axioms of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID)
The process algebra $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID is axiomatized by the axioms of $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID given in Definition 3.2.2 on page 69 *minus* Axiom DRTM1, *plus* Axioms DRTCM1–DRTCM5, DRTCM12–DRTCM13, and DRTCF1–DRTCF2 shown in Table 29, and Axioms DRTCM6–DRTCM7 shown in Table 30 on the next page: $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID = A1–A5 + DRT1–DRT5 + DCS1–DCS4 + DRTM2–DRTM6 + DRTCM1–DRTCM7 + DRTCM12–DRTCM13 + DRTCF1–DRTCF2.

$x \parallel y = x \mathbin{\|\!\_} y + y \mathbin{\|\!\_} x + x \mid y$		DRTCM1
$\underline{\underline{a}} \mid \underline{\underline{b}} \cdot x = (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot x$		DRTCM2
$\underline{\underline{a}} \cdot x \mid \underline{\underline{b}} = (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot x$		DRTCM3
$\underline{\underline{a}} \cdot x \mid \underline{\underline{b}} \cdot y = (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot (x \parallel y)$		DRTCM4
$\sigma_{\mathrm{rel}}(x) \mid \sigma_{\mathrm{rel}}(y) = \sigma_{\mathrm{rel}}(x \mid y)$		DRTCM5
$(x + y) \mid z = x \mid z + y \mid z$		DRTCM12
$x \mid (y + z) = x \mid y + x \mid z$		DRTCM13
$\underline{\underline{a}} \mid \underline{\underline{b}} = \underline{\underline{c}}$	if $\gamma(a,b) = c \neq \delta$	DRTCF1
$\underline{\underline{a}} \mid \underline{\underline{b}} = \underline{\underline{\delta}}$	if $\gamma(a,b) = \delta$	DRTCF2

Table 29: Axioms for the (communicating) merge.

Remark 3.4.4 (Axioms DRTCF1–DRTCF2)
Note that, by Definition 3.4.1, if we dropped the condition $c \neq \delta$ in DRTCF1, then DRTCF2 would be a special case of DRTCF1, and so DRTCF2 would not be needed anymore. However, for historical reasons we retain DRTCF1 and DRTCF2 as given in Table 29.

Definition 3.4.5 (Semantics of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID)
The semantics of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID are given by the term deduction system $T(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID})$ induced by the deduction rules for $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID given in Definition 3.2.3 on page 69 and the deduction rules for the (communicating) merge given in Table 31 on the next page.

$$\sigma_{\mathrm{rel}}(x) \mid \nu_{\mathrm{rel}}(y) = \underline{\underline{\delta}} \qquad \mathrm{DRTCM6}$$
$$\nu_{\mathrm{rel}}(x) \mid \sigma_{\mathrm{rel}}(y) = \underline{\underline{\delta}} \qquad \mathrm{DRTCM7}$$

Table 30: Additional axioms for $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID.

$$\frac{x \xrightarrow{a} x',\ y \xrightarrow{b} y',\ \gamma(a,b) = c}{x \parallel y \xrightarrow{c} x' \parallel y'} \qquad \frac{x \xrightarrow{a} x',\ y \xrightarrow{b} y',\ \gamma(a,b) = c}{x \mid y \xrightarrow{c} x' \parallel y'}$$

$$\frac{x \xrightarrow{a} x',\ y \xrightarrow{b} \surd,\ \gamma(a,b) = c}{x \parallel y \xrightarrow{c} x'} \qquad \frac{x \xrightarrow{a} x',\ y \xrightarrow{b} \surd,\ \gamma(a,b) = c}{x \mid y \xrightarrow{c} x'}$$

$$\frac{x \xrightarrow{a} \surd,\ y \xrightarrow{b} y',\ \gamma(a,b) = c}{x \parallel y \xrightarrow{c} y'} \qquad \frac{x \xrightarrow{a} \surd,\ y \xrightarrow{b} y',\ \gamma(a,b) = c}{x \mid y \xrightarrow{c} y'}$$

$$\frac{x \xrightarrow{a} \surd,\ y \xrightarrow{b} \surd,\ \gamma(a,b) = c}{x \parallel y \xrightarrow{c} \surd} \qquad \frac{x \xrightarrow{a} \surd,\ y \xrightarrow{b} \surd,\ \gamma(a,b) = c}{x \mid y \xrightarrow{c} \surd}$$

$$\frac{x \xrightarrow{\sigma} x',\ y \xrightarrow{\sigma} y'}{x \mid y \xrightarrow{\sigma} x' \mid y'}$$

Table 31: Deduction rules for the (communicating) merge.

Definition 3.4.6 (Bisimulation and Bisimulation Model for $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID)
Bisimulation for $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID and the corresponding bisimulation model are defined in the same way as for $\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$ and BPA respectively. Replace "$\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$" by "$\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID" in Definition 2.4.5 on page 14 and "BPA" by "$\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID" in Definition 2.2.11 on page 8.

Definition 3.4.7 (Basic Terms of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID)
If we speak of basic terms in the context of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID, we mean $(\sigma, \underline{\underline{\delta}})$-basic terms as defined in Definition 2.5.6 on page 23.

Definition 3.4.8 (Number of Symbols of an $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID Term)
We define $n(x)$, the number of symbols of x, inductively as follows:

(i). For $a \in A_\delta$, we define $n(\underline{\underline{a}}) = 1$,

(ii). for closed $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID terms x and y, we define $n(x + y) = n(x \cdot y) = n(x \parallel y) = n(x \mathbin{\lfloor\!\lfloor} y) = n(x \mid y) = n(x) + n(y) + 1$,

(iii). for a closed $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID term x, we define $n(\sigma_{\mathrm{rel}}(x)) = n(\nu_{\mathrm{rel}}(x)) = n(x) + 1$.

Theorem 3.4.9 (Elimination for $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID)
Let t be a closed $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID term. Then there is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term s such that $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID $\vdash s = t$.

Proof Let t be a closed $\mathrm{ACP}^-_{\mathrm{drt}}$-ID term. The theorem is proven by induction on $n(t)$ and case distinction on the general structure of t.

(i). $t \equiv \underline{\underline{a}}$ for some $a \in A_\delta$. Then t is a closed $\mathrm{BPA}^-_{\mathrm{drt}}$-ID term.

(ii). $t \equiv t_1 + t_2$ for closed $\mathrm{ACP}^-_{\mathrm{drt}}$-ID terms t_1 and t_2. By induction there are closed $\mathrm{BPA}^-_{\mathrm{drt}}$-ID terms s_1 and s_2 such that $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_1 = s_1$ and $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_2 = s_2$. But then also $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_1 + t_2 = s_1 + s_2$ and $s_1 + s_2$ is a closed $\mathrm{BPA}^-_{\mathrm{drt}}$-ID term.

(iii). $t \equiv t_1 \cdot t_2$ for closed $\mathrm{ACP}^-_{\mathrm{drt}}$-ID terms t_1 and t_2. This case is treated analogously to case (ii).

(iv). $t \equiv \sigma_{\mathrm{rel}}(t_1)$ for a closed $\mathrm{ACP}^-_{\mathrm{drt}}$-ID term t_1. This case is treated analogously to case (ii).

(v). $t \equiv \nu_{\mathrm{rel}}(t_1)$ for a closed $\mathrm{ACP}^-_{\mathrm{drt}}$-ID term t_1. This case is treated analogously to case (ii).

(vi). $t \equiv t_1 \mathbin{|\!\lfloor} t_2$ for closed $\mathrm{ACP}^-_{\mathrm{drt}}$-ID terms t_1 and t_2. By induction there are closed $\mathrm{BPA}^-_{\mathrm{drt}}$-ID terms s_1 and s_2 such that $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_1 = s_1$ and $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_2 = s_2$. By Theorem 2.5.12, the elimination theorem for $\mathrm{BPA}^-_{\mathrm{drt}}$-ID, there are basic terms r_1 and r_2 such that $\mathrm{BPA}^-_{\mathrm{drt}}\text{-ID} \vdash s_1 = r_1$ and $\mathrm{BPA}^-_{\mathrm{drt}}\text{-ID} \vdash s_2 = r_2$. But then also, $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_1 = r_1$, $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_2 = r_2$, and $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_1 \mathbin{|\!\lfloor} t_2 = r_1 \mathbin{|\!\lfloor} r_2$. We prove this case by induction on the structure of basic term r_1:

(a) $r_1 \equiv \underline{\underline{a}}$ for some $a \in A_\delta$. Then $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_1 \mathbin{|\!\lfloor} t_2 = r_1 \mathbin{|\!\lfloor} r_2 = \underline{\underline{a}} \mathbin{|\!\lfloor} r_2 = \underline{\underline{a}} \cdot r_2$, and $\underline{\underline{a}} \cdot r_2$ is a closed $\mathrm{BPA}^-_{\mathrm{drt}}$-ID term.

(b) $r_1 \equiv \underline{\underline{a}} \cdot r_1'$ for some $a \in A_\delta$ and basic term r_1'. Then $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_1 \mathbin{|\!\lfloor} t_2 = r_1 \mathbin{|\!\lfloor} r_2 = \underline{\underline{a}} \cdot r_1' \mathbin{|\!\lfloor} r_2 = \underline{\underline{a}} \cdot (r_1' \parallel r_2)$. By the induction hypothesis there exists a closed $\mathrm{BPA}^-_{\mathrm{drt}}$-ID term p such that $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash r_1' \parallel r_2 = p$. Then, $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_1 \mathbin{|\!\lfloor} t_2 = \underline{\underline{a}} \cdot (r_1' \parallel r_2) = \underline{\underline{a}} \cdot p$, and $\underline{\underline{a}} \cdot p$ is a closed $\mathrm{BPA}^-_{\mathrm{drt}}$-ID term.

(c) $r_1 \equiv r_1' + r_1''$ for basic terms r_1' and r_1''. Then $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_1 \mathbin{|\!\lfloor} t_2 = r_1 \mathbin{|\!\lfloor} r_2 = (r_1' + r_1'') \mathbin{|\!\lfloor} r_2 = r_1' \mathbin{|\!\lfloor} r_2 + r_1'' \mathbin{|\!\lfloor} r_2$. By induction there exist closed $\mathrm{BPA}^-_{\mathrm{drt}}$-ID terms p_1 and p_2 such that $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash r_1' \mathbin{|\!\lfloor} r_2 = p_1$ and $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash r_1'' \mathbin{|\!\lfloor} r_2 = p_2$. Then also $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_1 \mathbin{|\!\lfloor} t_2 = r_1' \mathbin{|\!\lfloor} r_2 + r_1'' \mathbin{|\!\lfloor} r_2 = p_1 + p_2$, and $p_1 + p_2$ is a closed $\mathrm{BPA}^-_{\mathrm{drt}}$-ID term.

(d) $r_1 \equiv \sigma_{\mathrm{rel}}(r_1')$ for a basic term r_1'. By Lemma 2.5.10 there is a basic term r_2' such that either $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash r_2 = \nu_{\mathrm{rel}}(r_2)$ or $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash r_2 = \nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')$ with $n(r_2') < n(r_2)$. With case analysis we obtain:

i. $r_2 = \nu_{\mathrm{rel}}(r_2)$. Then $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_1 \mathbin{|\!\lfloor} t_2 = r_1 \mathbin{|\!\lfloor} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{|\!\lfloor} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{|\!\lfloor} \nu_{\mathrm{rel}}(r_2) = \underline{\underline{\delta}}$, and $\underline{\underline{\delta}}$ is a closed $\mathrm{BPA}^-_{\mathrm{drt}}$-ID term.

ii. $r_2 = \nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')$ for a basic term r_2'. Then $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_1 \mathbin{|\!\lfloor} t_2 = r_1 \mathbin{|\!\lfloor} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{|\!\lfloor} t_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{|\!\lfloor} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{|\!\lfloor} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) = \sigma_{\mathrm{rel}}(r_1' \mathbin{|\!\lfloor} r_2')$. By the induction hypothesis there is a closed $\mathrm{BPA}^-_{\mathrm{drt}}$-ID term p such that $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash r_1' \mathbin{|\!\lfloor} r_2' = p$. But then also $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID} \vdash t_1 \mathbin{|\!\lfloor} t_2 = \sigma_{\mathrm{rel}}(r_1' \mathbin{|\!\lfloor} r_2') = \sigma_{\mathrm{rel}}(p)$, and $\sigma_{\mathrm{rel}}(p)$ is a closed $\mathrm{BPA}^-_{\mathrm{drt}}$-ID term.

(vii). $t \equiv t_1 \mid t_2$ for closed $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID terms t_1 and t_2. By induction there are closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID terms s_1 and s_2 such that $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 = s_1$ and $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_2 = s_2$. By Theorem 2.5.12, the elimination theorem for $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID, there are basic terms r_1 and r_2 such that $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash s_1 = r_1$ and $\mathrm{BPA}^{-}_{\mathrm{drt}}\text{–ID} \vdash s_2 = r_2$. But then also, $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 = r_1$, $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_2 = r_2$, and $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \mid t_2 = r_1 \mid r_2$. We prove this case by simultaneous induction on the structure of basic terms r_1 and r_2. We examine all possible cases:

(a) $r_1 \equiv \underline{\underline{a}}$ and $r_2 \equiv \underline{\underline{b}}$ for some $a, b \in A_\delta$. Suppose that $\gamma(a, b) = c$. Then we have $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \underline{\underline{a}} \mid \underline{\underline{b}} = \underline{\underline{c}}$, and $\underline{\underline{c}}$ is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

(b) $r_1 \equiv \underline{\underline{a}}$ and $r_2 \equiv \underline{\underline{b}} \cdot r_2'$ for some $a, b \in A_\delta$ and some basic term r_2'. Suppose that $\gamma(a, b) = c$. Then we have $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \underline{\underline{a}} \mid \underline{\underline{b}} \cdot r_2' = \underline{\underline{c}} \cdot r_2'$, and $\underline{\underline{c}} \cdot r_2'$ is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

(c) $r_1 \equiv \underline{\underline{a}} \cdot r_1'$ and $r_2 \equiv \underline{\underline{b}}$ for some $a, b \in A_\delta$ and some basic term r_1'. This case is treated symmetrically to the previous case.

(d) $r_1 \equiv \underline{\underline{a}} \cdot r_1'$ and $r_2 \equiv \underline{\underline{b}} \cdot r_2'$ for some $a, b \in A_\delta$ and some basic terms r_1' and r_2'. Suppose that $\gamma(a, b) = c$. Then we have $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \underline{\underline{a}} \cdot r_1' \mid \underline{\underline{b}} \cdot r_2' = \underline{\underline{c}} \cdot (r_1' \parallel r_2')$. By the induction hypothesis there exists a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term s' such that $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash r_1' \parallel r_2' = s'$. So $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \mid t_2 = \underline{\underline{c}} \cdot (r_1' \parallel r_2') = \underline{\underline{c}} \cdot s'$, and $\underline{\underline{c}} \cdot s'$ is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

(e) $r_1 \equiv r_1' + r_1''$ for some basic terms r_1' and r_1'', and r_2 is of arbitrary form. Then $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \mid t_2 = r_1 \mid r_2 = (r_1' + r_1'') \mid r_2 = r_1' \mid r_2 + r_1'' \mid r_2$. By the induction hypothesis there exist closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID terms p_1 and p_2 such that $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash r_1' \mid r_2 = p_1$ and $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash r_1'' \mid r_2 = p_2$. So, we have $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \mid t_2 = r_1' \mid r_2 + r_1'' \mid r_2 = p_1 + p_2$, and $p_1 + p_2$ is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

(f) r_1 is of arbitrary form and $r_2 \equiv r_2' + r_2''$ for some basic terms r_2' and r_2''. This case is treated symmetrically to the previous case.

(g) $r_1 \equiv \sigma_{\mathrm{rel}}(r_1')$ for some basic term r_1', and r_2 is of arbitrary form. By Lemma 2.5.10 there is a basic term r_2' such that either $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash r_2 = \nu_{\mathrm{rel}}(r_2)$ or $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash r_2 = \nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')$ with $n(r_2') < n(r_2)$. With case analysis we obtain:

   i. $r_2 = \nu_{\mathrm{rel}}(r_2)$. Then $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \sigma_{\mathrm{rel}}(r_1') \mid \nu_{\mathrm{rel}}(r_2) = \underline{\underline{\delta}}$, and $\underline{\underline{\delta}}$ is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

   ii. $r_2 = \nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')$ for a basic term r_2'. Then $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \sigma_{\mathrm{rel}}(r_1') \mid (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) = \sigma_{\mathrm{rel}}(r_1') \mid \sigma_{\mathrm{rel}}(r_2') = \sigma_{\mathrm{rel}}(r_1' \mid r_2')$. By the induction hypothesis there is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term p such that $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash r_1' \mid r_2' = p$. But then also $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \mid t_2 = \sigma_{\mathrm{rel}}(r_1' \mid r_2') = \sigma_{\mathrm{rel}}(p)$, and $\sigma_{\mathrm{rel}}(p)$ is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

(h) r_1 is of arbitrary form and $r_2 \equiv \sigma_{\mathrm{rel}}(r_2')$ for some basic term r_2'. This case is treated symmetrically to the previous case.

(viii). $t \equiv t_1 \parallel t_2$ for closed $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID terms t_1 and t_2. Then $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \parallel t_2 = t_1 \mathbin{⌊\!⌊} t_2 + t_2 \mathbin{⌊\!⌊} t_1 + t_1 \mid t_2$. By (vi) and (vii) there are closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID terms p_1, p_2, and p_3, such that $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \mathbin{⌊\!⌊} t_2 = p_1$, $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_2 \mathbin{⌊\!⌊} t_1 = p_2$, and $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \mid t_2 = p_3$. But then also $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID} \vdash t_1 \parallel t_2 = t_1 \mathbin{⌊\!⌊} t_2 + t_2 \mathbin{⌊\!⌊} t_1 + t_1 \mid t_2 = p_1 + p_2 + p_3$, and $p_1 + p_2 + p_3$ is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

■

Corollary 3.4.10 (Elimination for $\text{ACP}^-_{\text{drt}}$–ID)
Let t be a closed $\text{ACP}^-_{\text{drt}}$-ID term. Then there is a basic term s such that $\text{ACP}^-_{\text{drt}}\text{-ID} \vdash s = t$.

Proof This follows immediately from:

(i). The elimination theorem for $\text{ACP}^-_{\text{drt}}$–ID (see Theorem 3.4.9),

(ii). the elimination theorem for $\text{BPA}^-_{\text{drt}}$–ID (see Theorem 2.5.12),

(iii). the fact that all axioms of $\text{BPA}^-_{\text{drt}}$–ID are also contained in $\text{ACP}^-_{\text{drt}}$–ID.

■

Remark 3.4.11 (Elimination for $\text{ACP}^-_{\text{drt}}$–ID)
Elimination for $\text{ACP}^-_{\text{drt}}$–ID is also claimed (without proof) in Theorem 4.2 of [11].

Theorem 3.4.12 (Soundness of $\text{ACP}^-_{\text{drt}}$–ID)
The set of closed $\text{ACP}^-_{\text{drt}}$-ID terms modulo bisimulation equivalence is a model of $\text{ACP}^-_{\text{drt}}$-ID.

Proof We only prove soundness for the axioms which are added to $\text{BPA}^-_{\text{drt}}$–ID to obtain $\text{ACP}^-_{\text{drt}}$–ID.

Axioms DRTM2–DRTM6 The proofs for the soundness of these axioms with respect to PA^-_{drt}–ID that are given in Theorem 3.2.11 remain valid, since there are no new deduction rules dealing with $\mathbin{⌊\!⌊}$.

Axiom DRTCM1 Take the relation:

$$R = \{(s,s),(s \parallel t, t \parallel s),(s \parallel t, s \mathbin{⌊\!⌊} t + t \mathbin{⌊\!⌊} s + s \mid t) \mid s,t \in C(\text{ACP}^-_{\text{drt}}\text{-ID})\}$$

First we look at the transitions of the left-hand side:

(i). Suppose that $s \parallel t \xrightarrow{a} p$. First we look at the $(s \parallel t, t \parallel s)$ pairs. By inspection of the deduction rules we distinguish the following cases:

(a) $s \xrightarrow{a} p_1$ and $p \equiv p_1 \parallel t$. Then $t \parallel s \xrightarrow{a} t \parallel p_1$, and $(p_1 \parallel t, t \parallel p_1) \in R$
(b) $t \xrightarrow{a} p_2$ and $p \equiv s \parallel p_2$. Then $t \parallel s \xrightarrow{a} p_2 \parallel s$, and $(s \parallel p_2, p_2 \parallel s) \in R$.
(c) $s \xrightarrow{a} \surd$ and $p \equiv t$. Then $t \parallel s \xrightarrow{a} t$, and $(t,t) \in R$.
(d) $t \xrightarrow{a} \surd$ and $p \equiv s$. Then $t \parallel s \xrightarrow{a} s$, and $(s,s) \in R$.
(e) $s \xrightarrow{b} p_1$, $t \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_1 \parallel p_2$. Then $t \parallel s \xrightarrow{a} p_2 \parallel p_1$, and $(p_1 \parallel p_2, p_2 \parallel p_1) \in R$.
(f) $s \xrightarrow{b} \surd$, $t \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_2$. Then $t \parallel s \xrightarrow{a} p_2$, and $(p_2,p_2) \in R$.
(g) $s \xrightarrow{b} p_1$, $t \xrightarrow{c} \surd$, $\gamma(b,c) = a$, and $p \equiv p_1$. Then $t \parallel s \xrightarrow{a} p_1$, and $(p_1,p_1) \in R$.

We continue with the $(s \parallel t, s \mathbin{⌊\!⌊} t + t \mathbin{⌊\!⌊} s)$ pairs. Distinguishing the same cases as above, we derive:

(a) $s \mathbin{⌊\!⌊} t \xrightarrow{a} p_1 \parallel t$. Then $s \mathbin{⌊\!⌊} t + t \mathbin{⌊\!⌊} s + s \mid t \xrightarrow{a} p_1 \parallel t$, and $(p_1 \parallel t, t \parallel p_1) \in R$.
(b) $t \mathbin{⌊\!⌊} s \xrightarrow{a} p_2 \parallel s$. Then $s \mathbin{⌊\!⌊} t + t \mathbin{⌊\!⌊} s + s \mid t \xrightarrow{a} t \parallel p_2$, and $(s \parallel p_2, p_2 \parallel s) \in R$.
(c) $s \mathbin{⌊\!⌊} t \xrightarrow{a} t$. Then $s \mathbin{⌊\!⌊} t + t \mathbin{⌊\!⌊} s + s \mid t \xrightarrow{a} t$, and $(t,t) \in R$.

(d) $t \mathbin{\|\!\lfloor} s \xrightarrow{a} s$. Then $s \mathbin{\|\!\lfloor} t + t \mathbin{\|\!\lfloor} s + s \mid t \xrightarrow{a} s$, and $(s, s) \in R$.
(e) $s \mid t \xrightarrow{a} p_1 \parallel p_2$. Then $s \mathbin{\|\!\lfloor} t + t \mathbin{\|\!\lfloor} s + s \mid t \xrightarrow{a} p_1 \parallel p_2$, and $(p_1 \parallel p_2, p_1 \parallel p_2) \in R$.
(f) $s \mid t \xrightarrow{a} p_2$. Then $s \mathbin{\|\!\lfloor} t + t \mathbin{\|\!\lfloor} s + s \mid t \xrightarrow{a} p_2$, and $(p_2, p_2) \in R$.
(g) $s \mid t \xrightarrow{a} p_1$. Then $s \mathbin{\|\!\lfloor} t + t \mathbin{\|\!\lfloor} s + s \mid t \xrightarrow{a} p_1$, and $(p_1, p_1) \in R$.

(ii). Suppose that $s \parallel t \xrightarrow{a} \surd$. By inspection of the deduction rules we can conclude that $s \xrightarrow{b} \surd$, $t \xrightarrow{c} \surd$, and $\gamma(b, c) = a$. Therefore, $t \parallel s \xrightarrow{a} \surd$, and continuing, $s \mid t \xrightarrow{a} \surd$, so $s \mathbin{\|\!\lfloor} t + t \mathbin{\|\!\lfloor} s + s \mid t \xrightarrow{a} \surd$.

(iii). Suppose that $s \parallel t \xrightarrow{\sigma} p$. By inspection of the deduction rules we can conclude that $s \xrightarrow{\sigma} p_1$ and $t \xrightarrow{\sigma} p_2$ and $p \equiv p_1 \parallel p_2$. Therefore, $t \parallel s \xrightarrow{\sigma} p_2 \parallel p_1$, and note that $(p_1 \parallel p_2, p_2 \parallel p_1) \in R$. Continuing, we also have $s \mathbin{\|\!\lfloor} t \xrightarrow{\sigma} p_1 \mathbin{\|\!\lfloor} p_2$ $t \mathbin{\|\!\lfloor} s \xrightarrow{\sigma} p_2 \mathbin{\|\!\lfloor} p_1$, and $s \mid t \xrightarrow{\sigma} p_1 \mid p_2$. Therefore, $s \mathbin{\|\!\lfloor} t + t \mathbin{\|\!\lfloor} s + s \mid t \xrightarrow{\sigma} p_1 \mathbin{\|\!\lfloor} p_2 + p_2 \mathbin{\|\!\lfloor} p_1 + p_1 \mid p_2$, and note that $(p_1 \parallel p_2, p_1 \mathbin{\|\!\lfloor} p_2 + p_2 \mathbin{\|\!\lfloor} p_1 + p_1 \mid p_2) \in R$.

Secondly, we look at the transitions of the right-hand side:

(i). Suppose that $t \parallel s \xrightarrow{a} p$. This case is handled in the same way as the corresponding (sub)case for the left-hand side shown above.

(ii). Suppose that $t \parallel s \xrightarrow{a} \surd$. This case is handled in the same way as the corresponding (sub)case for the left-hand side shown above.

(iii). Suppose that $t \parallel s \xrightarrow{\sigma} p$. This case is handled in the same way as the corresponding (sub)case for the left-hand side shown above.

(iv). Suppose that $s \mathbin{\|\!\lfloor} t + t \mathbin{\|\!\lfloor} s + s \mid t \xrightarrow{a} p$. By inspection of the deduction rules we distinguish the following cases:
(a) $s \xrightarrow{a} p_1$ and $p \equiv p_1 \parallel t$. Then $s \parallel t \xrightarrow{a} p_1 \parallel t$, and $(p_1 \parallel t, p_1 \parallel t) \in R$.
(b) $t \xrightarrow{a} p_2$ and $p \equiv p_2 \parallel s$. Then $s \parallel t \xrightarrow{a} s \parallel p_2$, and $(s \parallel p_2, p_2 \parallel s) \in R$.
(c) $s \xrightarrow{a} \surd$ and $p \equiv t$. Then $s \parallel t \xrightarrow{a} t$, and $(t, t) \in R$.
(d) $t \xrightarrow{a} \surd$ and $p \equiv s$. Then $s \parallel t \xrightarrow{a} s$, and $(s, s) \in R$.
(e) $s \xrightarrow{b} p_1$, $t \xrightarrow{c} p_2$, $\gamma(b, c) = a$, and $p \equiv p_1 \parallel p_2$. Then $s \parallel t \xrightarrow{a} p_1 \parallel p_2$, and $(p_1 \parallel p_2, p_2 \parallel p_1) \in R$.
(f) $s \xrightarrow{b} \surd$, $t \xrightarrow{c} p_2$, $\gamma(b, c) = a$, and $p \equiv p_2$. Then $s \parallel t \xrightarrow{a} p_2$, and $(p_2, p_2) \in R$.
(g) $s \xrightarrow{b} p_1$, $t \xrightarrow{c} \surd$, $\gamma(b, c) = a$, and $p \equiv p_1$. Then $s \parallel t \xrightarrow{a} p_1$, and $(p_1, p_1) \in R$.

(v). Suppose that $s \mathbin{\|\!\lfloor} t + t \mathbin{\|\!\lfloor} s + s \mid t \xrightarrow{a} \surd$. By inspection of the deduction rules we can conclude that $s \xrightarrow{b} \surd$, $t \xrightarrow{c} \surd$, and $\gamma(b, c) = a$. Therefore, $s \parallel t \xrightarrow{a} \surd$.

(vi). Suppose that $s \mathbin{\|\!\lfloor} t + t \mathbin{\|\!\lfloor} s + s \mid t \xrightarrow{\sigma} p$. By inspection of the deduction rules we can conclude that $s \xrightarrow{\sigma} p_1$, $t \xrightarrow{\sigma} p_2$, and $p \equiv p_1 \mathbin{\|\!\lfloor} p_2 + p_2 \mathbin{\|\!\lfloor} p_1 + p_1 \mid p_2$. Since both s and t can perform a σ transition, we obtain $s \parallel t \xrightarrow{\sigma} p_1 \parallel p_2$, and note that $(p_1 \parallel p_2, p_1 \mathbin{\|\!\lfloor} p_2 + p_2 \mathbin{\|\!\lfloor} p_1 + p_1 \mid p_2) \in R$.

Axiom DRTCM2 Take the relation:

$$R = \{(s, s), (\underline{\underline{a}} \mid \underline{\underline{b}} \cdot s, (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot s) \mid s \in C(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID})\}$$

We look at the transitions of both sides at the same time. First, if $\gamma(a, b) = \delta$ there are no transitions possible on either side, and we are done. Otherwise, suppose $\gamma(a, b) = c$. Then the only possible transition on the left-hand side is $\underline{\underline{a}} \cdot s \mid \underline{\underline{b}} \xrightarrow{c} s$, and the only possible transition on the right-hand side is $(\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot s \xrightarrow{c} s$, and note that $(s, s) \in R$.

Axiom DRTCM3 Take the relation:

$$R = \{(s,s), (\underline{\underline{a}} \cdot s \mid \underline{\underline{b}}, (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot s) \mid s \in C(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID})\}$$

This axiom is treated symmetrically to the previous axiom.

Axiom DRTCM4 Take the relation:

$$R = \{(s,s), (\underline{\underline{a}} \cdot s \mid \underline{\underline{b}} \cdot t, (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot (s \parallel t)) \mid s,t \in C(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID})\}$$

We look at the transitions of both sides at the same time. First, if $\gamma(a,b) = \delta$ there are no transitions possible on either side, and we are done. Otherwise, suppose $\gamma(a,b) = c$. Then the only possible transition on the left-hand side is $\underline{\underline{a}} \cdot s \mid \underline{\underline{b}} \cdot t \xrightarrow{c} s \parallel t$, and the only possible transition on the right-hand side is $(\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot (s \parallel t) \xrightarrow{c} s \parallel t$, and note that $(s \parallel t, s \parallel t) \in R$.

Axiom DRTCM5 Take the relation:

$$R = \{(\sigma_{\mathrm{rel}}(s) \mid \sigma_{\mathrm{rel}}(t), \sigma_{\mathrm{rel}}(s \mid t)) \mid s,t \in C(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID})\}$$

We look at the transitions of both sides at the same time. The only possible transition of the left-hand side is $\sigma_{\mathrm{rel}}(s) \mid \sigma_{\mathrm{rel}}(t) \xrightarrow{\sigma} s \mid t$, and the only possible transition of the right-hand side is $\sigma_{\mathrm{rel}}(s \mid t) \xrightarrow{\sigma} s \mid t$, and note that $(s \mid t, s \mid t) \in R$.

Axiom DRTCM6 Take the relation:

$$R = \{(\sigma_{\mathrm{rel}}(s) \mid \nu_{\mathrm{rel}}(t), \underline{\underline{\delta}}) \mid s \in C(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID})\}$$

We look at the transitions of both sides at the same time. Observe that there are no transitions possible on the left-hand side: $\sigma_{\mathrm{rel}}(s) \mid \nu_{\mathrm{rel}}(t) \not\rightarrow$. Also for the right-hand side there are no transitions possible: $\underline{\underline{\delta}} \not\rightarrow$.

Axiom DRTCM7 Take the relation:

$$R = \{(\nu_{\mathrm{rel}}(x) \mid \sigma_{\mathrm{rel}}(y), \underline{\underline{\delta}})\}$$

This axiom is treated symmetrically to the previous axiom.

Axiom DRTCM12 Take the relation:

$$R = \{(s,s), ((s+t) \mid u, s \mid u + t \mid u) \mid s,t,u \in C(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID})\}$$

First we look at the transitions of the left-hand side.

(i). Suppose that $(s+t) \mid u \xrightarrow{a} p$. By inspection of the deduction rules we distinguish the following cases:

- (a) $s \xrightarrow{b} p_1$, $u \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_1 \parallel p_2$. Then $s \mid u \xrightarrow{a} p_1 \parallel p_2$, so also $s \mid u + t \mid u \xrightarrow{a} p_1 \parallel p_2$, and note that $(p_1 \parallel p_2, p_1 \parallel p_2) \in R$.
- (b) $t \xrightarrow{b} p_1$, $u \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_1 \parallel p_2$. This case is treated symmetrically to the previous case.
- (c) $s \xrightarrow{b} \surd$, $u \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_2$. Then $s \mid u \xrightarrow{a} p_2$, so also $s \mid u + t \mid u \xrightarrow{a} p_2$, and note that $(p_2, p_2) \in R$.

(d) $t \xrightarrow{b} \surd$, $u \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_2$. This case is treated symmetrically to the previous case.

(e) $s \xrightarrow{b} p_1$, $u \xrightarrow{c} \surd$, $\gamma(b,c) = a$, and $p \equiv p_1$. Then $s \mid u \xrightarrow{a} p_1$, so also $s \mid u + t \mid u \xrightarrow{a} p_1$, and note that $(p_1, p_1) \in R$.

(f) $t \xrightarrow{b} p_1$, $u \xrightarrow{c} \surd$, $\gamma(b,c) = a$, and $p \equiv p_1$. This case is treated symmetrically to the previous case.

(ii). Suppose that $(s + t) \mid u \xrightarrow{a} \surd$.

(a) $s \xrightarrow{b} \surd$, $u \xrightarrow{c} \surd$, and $\gamma(b,c) = a$. Then $s \mid u \xrightarrow{a} \surd$, so also $s \mid u + t \mid u \xrightarrow{a} \surd$.

(b) $t \xrightarrow{b} \surd$, $u \xrightarrow{c} \surd$, and $\gamma(b,c) = a$. This case is treated symmetrically to the previous case.

(iii). Suppose that $(s+t) \mid u \xrightarrow{\sigma} p$. By inspection of the deduction rules we distinguish the following cases:

(a) $s \xrightarrow{\sigma} p_1$, $t \stackrel{\sigma}{\nrightarrow}$, $u \xrightarrow{\sigma} p_2$, and $p \equiv p_1 \mid p_2$. Then $s \mid u \xrightarrow{\sigma} p_1 \mid p_2$ and $t \mid u \stackrel{\sigma}{\nrightarrow}$, so $s \mid u + t \mid u \xrightarrow{\sigma} p_1 \mid p_2$, and note that $(p_1 \mid p_2, p_1 \mid p_2) \in R$.

(b) $s \stackrel{\sigma}{\nrightarrow}$, $t \xrightarrow{\sigma} p_1$, $u \xrightarrow{\sigma} p_2$, and $p \equiv p_1 \mid p_2$. This case is treated symmetrically to the previous case.

(c) $s \xrightarrow{\sigma} p_1$, $t \xrightarrow{\sigma} p_2$, $u \xrightarrow{\sigma} p_3$, and $p \equiv (p_1 + p_2) \mid p_3$. Then $s \mid u \xrightarrow{\sigma} p_1 \mid p_3$ and $t|u \xrightarrow{\sigma} p_2|p_3$, so $s|u+t|u \xrightarrow{\sigma} p_1|p_3+p_2|p_3$, and note that $((p_1+p_2)|p_3, p_1|p_3+p_2 \mid p_3) \in R$.

Secondly, we look at the transitions of the right-hand side:

(i). Suppose that $s|u+t|u \xrightarrow{a} p$. By inspection of the deduction rules we distinguish the following cases:

(a) $s \xrightarrow{b} p_1$, $u \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_1 \parallel p_2$. Then $s + t \xrightarrow{b} p_1$, so $(s + t) \mid u \xrightarrow{a} p_1 \parallel p_2$, and note that $(p_1 \parallel p_2, p_1 \parallel p_2) \in R$.

(b) $t \xrightarrow{b} p_1$, $u \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_1 \parallel p_2$. This case is treated symmetrically to the previous case.

(c) $s \xrightarrow{b} \surd$, $u \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_2$. Then $s + t \xrightarrow{b} \surd$, so $(s + t) \mid u \xrightarrow{a} p_2$, and note that (p_2, p_2).

(d) $t \xrightarrow{b} \surd$, $u \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_2$. This case is treated symmetrically to the previous case.

(e) $s \xrightarrow{b} p_1$, $u \xrightarrow{c} \surd$, $\gamma(b,c) = a$, and $p \equiv p_1$. Then $s + t \xrightarrow{b} p_1$, so $(s + t) \mid u \xrightarrow{a} p_1$, and note that (p_1, p_1).

(f) $t \xrightarrow{b} p_1$, $u \xrightarrow{c} \surd$, $\gamma(b,c) = a$, and $p \equiv p_1$. This case is treated symmetrically to the previous case.

(ii). Suppose that $s|u+t|u \xrightarrow{a} \surd$. By inspection of the deduction rules we distinguish the following cases:

(a) $s \xrightarrow{b} \surd$, $u \xrightarrow{c} \surd$, and $\gamma(b,c) = a$. Then $s + t \xrightarrow{b} \surd$, so $(s + t) \mid u \xrightarrow{a} \surd$.

(b) $t \xrightarrow{b} \surd$, $u \xrightarrow{c} \surd$, and $\gamma(b,c) = a$. This case is treated symmetrically to the previous case.

(iii). Suppose that $s|u+t|u \xrightarrow{\sigma} p$. By inspection of the deduction rules we distinguish the following cases:

(a) $s \xrightarrow{\sigma} p_1$, $t \not\xrightarrow{\sigma}$, $u \xrightarrow{\sigma} p_2$, and $p \equiv p_1 \mid p_2$. Then $s + t \xrightarrow{\sigma} p_1$, so $(s+t) \mid u \xrightarrow{\sigma} p_1 \mid p_2$, and note that $(p_1 \mid p_2, p_1 \mid p_2) \in R$.
(b) $s \not\xrightarrow{\sigma}$, $t \xrightarrow{\sigma} p_1$, $u \xrightarrow{\sigma} p_2$, and $p \equiv p_1 \mid p_2$. This case is treated symmetrically to the previous case.
(c) $s \xrightarrow{\sigma} p_1$, $t \xrightarrow{\sigma} p_2$, $u \xrightarrow{\sigma} p_3$, and $p \equiv p_1 \mid p_3 + p_2 \mid p_3$. Then $s + t \xrightarrow{\sigma} p_1 + p_2$, so $(s+t) \mid u \xrightarrow{\sigma} (p_1 + p_2) \mid p_3$, and note that $((p_1 + p_2) \mid p_3, p_1 \mid p_3 + p_2 \mid p_3) \in R$.

Axiom DRTCM13 Take the relation:

$$R = \{(s,s), (s \mid (t+u), s \mid t + s \mid u) \mid s,t,u \in C(\mathrm{ACP}^-_{\mathrm{drt}}\text{–ID})\}$$

This axiom is treated symmetrically to the previous axiom.

Axiom DRTCF1 Take the relation:

$$R = \{(\underline{\underline{a}} \mid \underline{\underline{b}}, \underline{\underline{c}})\}$$

We look at the transitions of both sides at the same time. By the definition of Axiom DRTCF1 we have that $\gamma(a,b) = c \neq \delta$. Then the only possible transition of the left-hand side is $\underline{\underline{a}} \mid \underline{\underline{b}} \xrightarrow{c} \surd$, and the only possible transition of the right-hand side is $\underline{\underline{c}} \xrightarrow{c} \surd$.

Axiom DRTCF2 Take the relation:

$$R = \{(\underline{\underline{a}} \mid \underline{\underline{b}}, \underline{\underline{\delta}})\}$$

We look at the transitions of both sides at the same time. As, by the definition of Axiom DRTCF2, we have that $\gamma(a,b) = \delta$, there are no transitions possible on either side.

■

Remark 3.4.13 (Soundness of $\mathrm{ACP}^-_{\mathrm{drt}}$–ID)
Soundness of $\mathrm{ACP}^-_{\mathrm{drt}}$–ID is also claimed (without proof) in Theorem 4.2 of [11].

Theorem 3.4.14 (Conservativity of $\mathrm{ACP}^-_{\mathrm{drt}}$–ID with respect to $\mathrm{BPA}^-_{\mathrm{drt}}$–ID)
The equational specification $\mathrm{ACP}^-_{\mathrm{drt}}$–ID is a conservative extension of the equational specification $\mathrm{BPA}^-_{\mathrm{drt}}$–ID.

Proof In order to prove conservativity it is sufficient to verify that the following conditions are satisfied:

(i). Bisimulation equivalence is definable in terms of predicate and relation symbols only,

(ii). $\mathrm{BPA}^-_{\mathrm{drt}}$–ID is a complete axiomatization with respect to the bisimulation equivalence model induced by $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID})$ (see Theorem 2.5.17),

(iii). $\mathrm{ACP}^-_{\mathrm{drt}}$–ID is a sound axiomatization with respect to the bisimulation equivalence model induced by $T(\mathrm{ACP}^-_{\mathrm{drt}}\text{–ID})$ (see Theorem 3.4.12),

(iv). $T(\mathrm{ACP}^-_{\mathrm{drt}}\text{–ID})$ is an operationally conservative extension of $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID})$.

And in order for $T(\mathrm{ACP}^-_{\mathrm{drt}}\text{–ID})$ indeed to be an operationally conservative extension of $T(\mathrm{BPA}^-_{\mathrm{drt}}\text{–ID})$ we must verify the following conditions:

(i). $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-ID})$ is a pure, well-founded term deduction system in path format,

(ii). $T(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{-ID})$ is a term deduction system in path format,

(iii). $T(\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-ID}) \oplus T(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{-ID})$ is defined.

That the above properties hold can be trivially checked from the relevant definitions. ■

Theorem 3.4.15 (Completeness of ACP$^{-}_{\mathrm{drt}}$–ID)
The equational specification ACP$^{-}_{drt}$-ID is a complete axiomatization of the set of closed ACP$^{-}_{drt}$-ID terms modulo bisimulation equivalence.

Proof By Verhoef's General Completeness Theorem (see [25], or Theorem 2.4.26 of [13]) this follows immediately from:

(i). ACP$^{-}_{\mathrm{drt}}$–ID has the elimination property for BPA$^{-}_{\mathrm{drt}}$–ID (see Theorem 3.4.9),

(ii). ACP$^{-}_{\mathrm{drt}}$–ID is a conservative extension of BPA$^{-}_{\mathrm{drt}}$–ID (see Theorem 3.4.14).

■

Remark 3.4.16 (Completeness of ACP$^{-}_{\mathrm{drt}}$–ID)
Completeness of ACP$^{-}_{\mathrm{drt}}$–ID is also claimed (without proof) in Theorem 4.2 of [11].

3.5 Soundness and Completeness of ACP$^{-}_{\mathrm{drt}}$–ID$'$

Definition 3.5.1 (Signature of ACP$^{-}_{\mathrm{drt}}$–ID$'$)
The signature of ACP$^{-}_{\mathrm{drt}}$–ID$'$ is identical to the signature of ACP$^{-}_{\mathrm{drt}}$–ID as given in Definition 3.4.2; it consists of the *undelayable atomic actions* $\{\underline{\underline{a}} | a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *time unit delay operator* σ_{rel}, the *"now" operator* ν_{rel}, the *(communicating) merge operator* $\|$, the *left merge operator* $⌊\!⌊$, and the *communication merge operator* $|$.

Definition 3.5.2 (Axioms of ACP$^{-}_{\mathrm{drt}}$–ID$'$)
The process algebra ACP$^{-}_{\mathrm{drt}}$–ID$'$ is axiomatized by the axioms of PA$^{-}_{\mathrm{drt}}$–ID$'$ given in Definition 3.3.2 on page 76 *minus* Axiom DRTM1, *plus* Axioms DRTCM1–DRTCM5, DRTCM12–DRTCM13, and DRTCF1–DRTCF2 shown in Table 29 on page 84, and Axioms DRTCM8–DRTCM11 shown in Table 32 on the next page: ACP$^{-}_{\mathrm{drt}}$–ID$'$ = A1–A5 + DRT1–DRT5 + DCS1–DCS4 + DRTM2–DRTM4, DRTM7–DRTM11 + DRTCM1–DRTCM5 + DRTCM8–13 + DRTCF1–DRTCF2.

Definition 3.5.3 (Semantics of ACP$^{-}_{\mathrm{drt}}$–ID$'$)
The semantics of ACP$^{-}_{\mathrm{drt}}$–ID$'$ are given by the term deduction system $T(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{-ID}')$ which is identical to the term deduction system $T(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{-ID})$ given in Definition 3.4.5 on page 84.

Definition 3.5.4 (Bisimulation and Bisimulation Model for ACP$^{-}_{\mathrm{drt}}$–ID$'$)
Bisimulation for ACP$^{-}_{\mathrm{drt}}$–ID$'$ and the corresponding bisimulation model are defined in the same way as for BPA$^{-}_{\mathrm{drt}}$–δ and BPA respectively. Replace "BPA$^{-}_{\mathrm{drt}}$–δ" by "ACP$^{-}_{\mathrm{drt}}$–ID$'$" in Definition 2.4.5 on page 14 and "BPA" by "ACP$^{-}_{\mathrm{drt}}$–ID$'$" in Definition 2.2.11 on page 8.

$$\underline{\underline{a}} \mid \sigma_{\mathrm{rel}}(x) = \underline{\underline{\delta}} \qquad \text{DRTCM8}$$
$$\sigma_{\mathrm{rel}}(x) \mid \underline{\underline{a}} = \underline{\underline{\delta}} \qquad \text{DRTCM9}$$
$$\underline{\underline{a}} \cdot x \mid \sigma_{\mathrm{rel}}(y) = \underline{\underline{\delta}} \qquad \text{DRTCM10}$$
$$\sigma_{\mathrm{rel}}(x) \mid \underline{\underline{a}} \cdot y = \underline{\underline{\delta}} \qquad \text{DRTCM11}$$

Table 32: Additional axioms for $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′.

Definition 3.5.5 (Basic Terms of $\mathbf{ACP}^{-}_{\mathbf{drt}}$–ID′)
If we speak of basic terms in the context of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′, we mean $(\sigma, \underline{\underline{\delta}})$-basic terms as defined in Definition 2.5.6 on page 23.

Definition 3.5.6 (Number of Symbols of an $\mathbf{ACP}^{-}_{\mathbf{drt}}$–ID′ Term)
We define $n(x)$, the number of symbols of x, inductively as follows:

(i). For $a \in A_\delta$, we define $n(\underline{\underline{a}}) = 1$,

(ii). for closed $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′ terms x and y, we define $n(x + y) = n(x \cdot y) = n(x \parallel y) = n(x \mathbin{⫽} y) = n(x \mid y) = n(x) + n(y) + 1$,

(iii). for a closed $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′ term x, we define $n(\sigma_{\mathrm{rel}}(x)) = n(\nu_{\mathrm{rel}}(x)) = n(x) + 1$.

Theorem 3.5.7 (Elimination for $\mathbf{ACP}^{-}_{\mathbf{drt}}$–ID′)
Let t be a closed $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′ term. Then there is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term s such that $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID}' \vdash s = t$.

Proof First a term rewriting system is given which is proven to be strongly normalizing. Thereafter, it is shown that every normal form of a closed $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′ term is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term.

The term rewriting system used is given in Table 33 on the following page. Note that we have added natural number subscripts n to the merge operators, in order to deal with the mutually recursive nature of definition of these operators. For a description of this technique, and a rigorous formal justification of its use, see Theorem 3.2.3 of [13] and the references given there. Using the method of the lexicographical path ordering we prove that the term rewriting system associated with $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′ is strongly normalizing. Thereto, the operator $\cdot$ is assigned the lexicographical status of the first argument, and the following well-founded ordering on constant and function symbols is used:

$$\underline{\underline{a}} < \sigma_{\mathrm{rel}} < + < \cdot < \mathbin{⫽}_2, \mid_2 < \parallel_2 < \mathbin{⫽}_3, \mid_3 < \cdots < \mathbin{⫽}_n, \mid_n < \parallel_n < \mathbin{⫽}_{n+1}, \mid_{n+1} < \ldots$$

That the left-hand side of every rewriting rule is bigger than the right-hand side with respect to the ordering $\succ_{\mathrm{lpo}}$, is shown by the following reductions:

$$\begin{aligned} x \parallel_n y &\succ_{\mathrm{lpo}} x \parallel_n^\star y \succ_{\mathrm{lpo}} x \parallel_n^\star y + x \parallel_n^\star y \succ_{\mathrm{lpo}} x \parallel_n^\star y + x \parallel_n^\star y + x \parallel_n^\star y \\ &\succ_{\mathrm{lpo}} (x \parallel_n^\star y) \mathbin{⫽}_n (x \parallel_n^\star y) + (x \parallel_n^\star y) \mathbin{⫽}_n (x \parallel_n^\star y) + (x \parallel_n^\star y) \mid_n (x \parallel_n^\star y) \\ &\succ_{\mathrm{lpo}} x \mathbin{⫽}_n y + y \mathbin{⫽}_n x + x \mid_n y \end{aligned}$$

$$
\begin{array}{rll}
x \parallel_n y \to x \mathbin{\lfloor\!\lfloor}_n y + y \mathbin{\lfloor\!\lfloor}_n x + x \mid_n y & & \text{RDRTCM1} \\
\underline{\underline{a}} \mathbin{\lfloor\!\lfloor}_n x \to \underline{\underline{a}} \cdot x & & \text{RDRTM2} \\
\underline{\underline{a}} \cdot x \mathbin{\lfloor\!\lfloor}_{n+1} y \to \underline{\underline{a}} \cdot (x \parallel_n y) & & \text{RDRTM3} \\
(x + y) \mathbin{\lfloor\!\lfloor}_n z \to x \mathbin{\lfloor\!\lfloor}_n z + y \mathbin{\lfloor\!\lfloor}_n z & & \text{RDRTM4} \\
\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n \underline{\underline{a}} \to \underline{\underline{\delta}} & & \text{RDRTM7} \\
\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n \underline{\underline{a}} \cdot y \to \underline{\underline{\delta}} & & \text{RDRTM8} \\
\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n (\underline{\underline{a}} + y) \to \sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n y & & \text{RDRTM9} \\
\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n (\underline{\underline{a}} \cdot y + z) \to \sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n z & & \text{RDRTM10} \\
\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\lfloor}_n \sigma_{\mathrm{rel}}(y) \to \sigma_{\mathrm{rel}}(x \mathbin{\lfloor\!\lfloor}_n y) & & \text{RDRTM11} \\
\underline{\underline{a}} \mid_n \underline{\underline{b}} \to \underline{\underline{c}} & \text{if } \gamma(a,b) = c \neq \delta & \text{RDRTCF1} \\
\underline{\underline{a}} \mid_n \underline{\underline{b}} \to \underline{\underline{\delta}} & \text{if } \gamma(a,b) = \delta & \text{RDRTCF2} \\
\underline{\underline{a}} \mid_n \underline{\underline{b}} \cdot x \to (\underline{\underline{a}} \mid_n \underline{\underline{b}}) \cdot x & & \text{RDRTCM2} \\
\underline{\underline{a}} \cdot x \mid_n \underline{\underline{b}} \to (\underline{\underline{a}} \mid_n \underline{\underline{b}}) \cdot x & & \text{RDRTCM3} \\
\underline{\underline{a}} \cdot x \mid_{n+1} \underline{\underline{b}} \cdot y \to (\underline{\underline{a}} \mid_{n+1} \underline{\underline{b}}) \cdot (x \parallel_n y) & & \text{RDRTCM4} \\
\sigma_{\mathrm{rel}}(x) \mid_n \sigma_{\mathrm{rel}}(y) \to \sigma_{\mathrm{rel}}(x \mid_n y) & & \text{RDRTCM5} \\
\underline{\underline{a}} \mid_n \sigma_{\mathrm{rel}}(x) \to \underline{\underline{\delta}} & & \text{RDRTCM8} \\
\sigma_{\mathrm{rel}}(x) \mid_n \underline{\underline{a}} \to \underline{\underline{\delta}} & & \text{RDRTCM9} \\
\underline{\underline{a}} \cdot x \mid_n \sigma_{\mathrm{rel}}(y) \to \underline{\underline{\delta}} & & \text{RDRTCM10} \\
\sigma_{\mathrm{rel}}(x) \mid_n \underline{\underline{a}} \cdot y \to \underline{\underline{\delta}} & & \text{RDRTCM11} \\
(x + y) \mid_n z \to x \mid_n z + y \mid_n z & & \text{RDRTCM12} \\
x \mid_n (y + z) \to x \mid_n y + x \mid_n z & & \text{RDRTCM13}
\end{array}
$$

Table 33: Term rewriting system for $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID}'$.

$$
\begin{array}{rl}
\underline{\underline{a}} \mid_n \underline{\underline{b}} & \succ_{\mathrm{lpo}} \underline{\underline{a}} \mid^{\star}_n \underline{\underline{b}} \\
& \succ_{\mathrm{lpo}} \underline{\underline{c}} \\
\underline{\underline{a}} \mid_n \underline{\underline{b}} & \succ_{\mathrm{lpo}} \underline{\underline{a}} \mid^{\star}_n \underline{\underline{b}} \\
& \succ_{\mathrm{lpo}} \underline{\underline{\delta}} \\
\underline{\underline{a}} \mid_n \underline{\underline{b}} \cdot x & \succ_{\mathrm{lpo}} \underline{\underline{a}} \mid^{\star}_n \underline{\underline{b}} \cdot x \succ_{\mathrm{lpo}} (\underline{\underline{a}} \mid^{\star}_n \underline{\underline{b}} \cdot x) \cdot (\underline{\underline{a}} \mid^{\star}_n \underline{\underline{b}} \cdot x) \succ_{\mathrm{lpo}} (\underline{\underline{a}} \mid_n \underline{\underline{b}} \cdot^{\star} x) \cdot (\underline{\underline{b}} \cdot x) \\
& \succ_{\mathrm{lpo}} (\underline{\underline{a}} \mid_n \underline{\underline{b}}) \cdot (\underline{\underline{b}} \cdot x) \succ_{\mathrm{lpo}} (\underline{\underline{a}} \mid_n \underline{\underline{b}}) \cdot^{\star} (\underline{\underline{b}} \cdot x) \succ_{\mathrm{lpo}} (\underline{\underline{a}} \mid_n \underline{\underline{b}}) \cdot (\underline{\underline{b}} \cdot^{\star} x) \\
& \succ_{\mathrm{lpo}} (\underline{\underline{a}} \mid_n \underline{\underline{b}}) \cdot x \\
\underline{\underline{a}} \cdot x \mid_n \underline{\underline{b}} & \succ_{\mathrm{lpo}} \underline{\underline{a}} \cdot x \mid^{\star}_n \underline{\underline{b}} \succ_{\mathrm{lpo}} (\underline{\underline{a}} \cdot x \mid^{\star}_n \underline{\underline{b}}) \cdot (\underline{\underline{a}} \cdot x \mid^{\star}_n \underline{\underline{b}}) \succ_{\mathrm{lpo}} (\underline{\underline{a}} \cdot^{\star} x \mid_n \underline{\underline{b}}) \cdot (\underline{\underline{a}} \cdot x) \\
& \succ_{\mathrm{lpo}} (\underline{\underline{a}} \mid_n \underline{\underline{b}}) \cdot (\underline{\underline{a}} \cdot x) \succ_{\mathrm{lpo}} (\underline{\underline{a}} \mid_n \underline{\underline{b}}) \cdot^{\star} (\underline{\underline{a}} \cdot x) \succ_{\mathrm{lpo}} (\underline{\underline{a}} \mid_n \underline{\underline{b}}) \cdot (\underline{\underline{a}} \cdot^{\star} x) \\
& \succ_{\mathrm{lpo}} (\underline{\underline{a}} \mid_n \underline{\underline{b}}) \cdot x
\end{array}
$$

$$
\begin{aligned}
\underline{\underline{a}} \cdot x \mid_{n+1} \underline{\underline{b}} \cdot y &\succ_{\mathrm{lpo}} \underline{\underline{a}} \cdot x \mid^{\star}_{n+1} \underline{\underline{b}} \cdot y \succ_{\mathrm{lpo}} (\underline{\underline{a}} \cdot x \mid^{\star}_{n+1} \underline{\underline{b}} \cdot y) \cdot (\underline{\underline{a}} \cdot x \mid^{\star}_{n+1} \underline{\underline{b}} \cdot y) \\
&\succ_{\mathrm{lpo}} (\underline{\underline{a}} \cdot x \mid^{\star}_{n+1} \underline{\underline{b}} \cdot y) \cdot ((\underline{\underline{a}} \cdot x \mid^{\star}_{n+1} \underline{\underline{b}} \cdot y) \ \|_n \ (\underline{\underline{a}} \cdot x \mid^{\star}_{n+1} \underline{\underline{b}} \cdot y)) \\
&\succ_{\mathrm{lpo}} (\underline{\underline{a}} \cdot^{\star} x \mid_{n+1} \underline{\underline{b}} \cdot^{\star} y) \cdot ((\underline{\underline{a}} \cdot x \mid^{\star}_{n+1} \underline{\underline{b}} \cdot y) \ \|_n \ (\underline{\underline{a}} \cdot x \mid^{\star}_{n+1} \underline{\underline{b}} \cdot y)) \\
&\succ_{\mathrm{lpo}} (\underline{\underline{a}} \mid_{n+1} \underline{\underline{b}}) \cdot (\underline{\underline{a}} \cdot x \ \|_n \ \underline{\underline{b}} \cdot y) \succ_{\mathrm{lpo}} (\underline{\underline{a}} \mid_{n+1} \underline{\underline{b}}) \cdot (\underline{\underline{a}} \cdot^{\star} x \ \|_n \ \underline{\underline{b}}^{\star} \cdot y) \\
&\succ_{\mathrm{lpo}} (\underline{\underline{a}} \mid_{n+1} \underline{\underline{b}}) \cdot (x \ \|_n \ y) \\
\sigma_{\mathrm{rel}}(x) \mid_n \sigma_{\mathrm{rel}}(y) &\succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x) \mid^{\star}_n \sigma_{\mathrm{rel}}(y) \succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x) \mid^{\star}_n \sigma_{\mathrm{rel}}(y)) \\
&\succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(\sigma^{\star}_{\mathrm{rel}}(x) \mid_n \sigma^{\star}_{\mathrm{rel}}(y)) \succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x \mid_n y) \\
\underline{\underline{a}} \mid_n \sigma_{\mathrm{rel}}(x) &\succ_{\mathrm{lpo}} \underline{\underline{a}} \mid^{\star}_n \sigma_{\mathrm{rel}}(x) \\
&\succ_{\mathrm{lpo}} \underline{\underline{\delta}} \\
\sigma_{\mathrm{rel}}(x) \mid_n \underline{\underline{a}} &\succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x) \mid^{\star}_n \underline{\underline{a}} \\
&\succ_{\mathrm{lpo}} \underline{\underline{\delta}} \\
\underline{\underline{a}} \cdot x \mid_n \sigma_{\mathrm{rel}}(y) &\succ_{\mathrm{lpo}} \underline{\underline{a}} \cdot x \mid^{\star}_n \sigma_{\mathrm{rel}}(y) \\
&\succ_{\mathrm{lpo}} \underline{\underline{\delta}} \\
\sigma_{\mathrm{rel}}(x) \mid_n \underline{\underline{a}} \cdot y &\succ_{\mathrm{lpo}} \sigma_{\mathrm{rel}}(x) \mid^{\star}_n \underline{\underline{a}} \cdot y \\
&\succ_{\mathrm{lpo}} \underline{\underline{\delta}} \\
(x + y) \mid_n z &\succ_{\mathrm{lpo}} (x + y) \mid^{\star}_n z \succ_{\mathrm{lpo}} (x + y) \mid^{\star}_n z + (x + y) \mid^{\star}_n z \\
&\succ_{\mathrm{lpo}} (x +^{\star} y) \mid_n z + (x +^{\star} y) \mid_n z \succ_{\mathrm{lpo}} (x \mid_n z) + (y \mid_n z) \\
x \mid_n (y + z) &\succ_{\mathrm{lpo}} x \mid^{\star}_n (y + z) \succ_{\mathrm{lpo}} x \mid^{\star}_n (y + z) + x \mid^{\star}_n (y + z) \\
&\succ_{\mathrm{lpo}} x \mid_n (y +^{\star} z) + x \mid_n (y +^{\star} z) \succ_{\mathrm{lpo}} x \mid_n (y) + x \mid_n (z)
\end{aligned}
$$

Note that we do not give reductions for RDRTM2–RDRTM11, as these already have been given in the proof of Theorem 3.5.7, and since the new ordering is a proper extension of the old one, these proofs remain valid.

It remains to prove that every normal form of a closed $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′ term is a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term. We prove this as follows: suppose that s is a normal form of a closed $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′ term, and furthermore suppose that s is not a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term. Now consider the smallest subterm s' of s that is not a closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID term. Then, s' must be of the form $s' \equiv s_1 \parallel s_2$, of the form $s' \equiv s_1 \mathbin{\|\!\lfloor} s_2$, or of the form $s' \equiv s_1 \mid s_2$, for closed $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID terms s_1 and s_2. By the elimination theorem for $\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID, Theorem 2.6.12, we may assume that s_1 and s_2 are basic terms. Now in the first case, $s' \equiv s_1 \parallel s_2$, clearly rewriting rule RDRTCM1 is applicable. This contradicts the assumption that s is a normal form, so this case cannot occur That the second case, $s' \equiv s_1 \mathbin{\|\!\lfloor} s_2$, cannot occur is proven in the same way as already done in the proof of the elimination theorem for $\mathrm{PA}^{-}_{\mathrm{drt}}$–ID′, Theorem 3.3.7. So it remains to derive a contradiction for the third case, $s' \equiv s_1 \mid s_2$. The following cases can be considered for basic terms s_1 and s_2 (for some $a, b \in A_\delta$ and basic terms s_1', s_1'', s_2', s_2''):

(i). If $s_1 \equiv \underline{\underline{a}}$ and $s_2 \equiv \underline{\underline{b}}$ we can apply RDRTCF1 if $\gamma(a, b) \neq \delta$, if not, we can apply RDRTCF2.

(ii). If $s_1 \equiv \underline{\underline{a}}$ and $s_2 \equiv \underline{\underline{b}} \cdot s_2'$, we can apply RDRTCM2.

(iii). If $s_1 \equiv \underline{\underline{a}}$ and $s_2 \equiv \sigma_{\mathrm{rel}}(s_2')$, we can apply RDRTCM8.

(iv). If $s_1 \equiv \underline{\underline{a}} \cdot s_1'$ and $s_2 \equiv \underline{\underline{b}}$, we can apply RDRTCM3.

(v). If $s_1 \equiv \underline{\underline{a}} \cdot s_1'$ and $s_2 \equiv \underline{\underline{b}} \cdot s_2'$, we can apply RDRTCM4.

(vi). If $s_1 \equiv \underline{\underline{a}} \cdot s_1'$ and $s_2 \equiv \sigma_{\mathrm{rel}}(s_2')$, we can apply RDRTCM10.

(vii). If $s_1 \equiv \sigma_{\mathrm{rel}}(s_1')$ and $s_2 \equiv \underline{\underline{b}}$, we can apply RDRTCM9.

(viii). If $s_1 \equiv \sigma_{\mathrm{rel}}(s_1')$ and $s_2 \equiv \underline{\underline{b}} \cdot s_2'$, we can apply RDRTCM11.

(ix). If $s_1 \equiv \sigma_{\mathrm{rel}}(s_1')$ and $s_2 \equiv \sigma_{\mathrm{rel}}(s_2')$, we can apply RDRTCM5.

(x). If $s_1 \equiv s_1' + s_1''$ and s_2 is of an arbitrary form, we can apply RDRTCM12.

(xi). If s_1 is of an arbitrary form and $s_2 \equiv s_2' + s_2''$, we can apply RDRTCM13.

This sums up all possible sixteen cases (with seven cases thrown together in (x). and (xi).). In all of these cases we can apply one of the rewriting rules, so s' is not a normal form. This contradicts the assumption that s is a normal form. From this contradiction we conclude that s does not contain a merge operator. Therefore s must be closed $\mathrm{BPA}^-_{\mathrm{drt}}$–ID term, which had to be proven. ■

Corollary 3.5.8 (Elimination for $\mathbf{ACP}^-_{\mathbf{drt}}$–ID′)
Let t be a closed ACP^-_{drt}-ID′ term. Then there is a basic term s such that ACP^-_{drt}-ID′ $\vdash s = t$.

Proof This follows immediately from:

(i). The elimination theorem for $\mathrm{ACP}^-_{\mathrm{drt}}$–ID′ (see Theorem 3.5.7),

(ii). the elimination theorem for $\mathrm{BPA}^-_{\mathrm{drt}}$–ID (see Theorem 2.5.12),

(iii). the fact that all axioms of $\mathrm{BPA}^-_{\mathrm{drt}}$–ID are also contained in $\mathrm{ACP}^-_{\mathrm{drt}}$–ID′.

■

Remark 3.5.9 (Elimination for $\mathbf{ACP}^-_{\mathbf{drt}}$–ID′)
Elimination for a slightly different version of $\mathrm{ACP}^-_{\mathrm{drt}}$–ID′ is also claimed (without proof) in Theorem 3.6.4 of [13] (where $\mathrm{ACP}^-_{\mathrm{drt}}$–ID′ is called $\mathrm{ACP}_{\mathrm{dt}}$).

Theorem 3.5.10 (Ground Equivalence of $\mathbf{ACP}^-_{\mathbf{drt}}$–ID and $\mathbf{ACP}^-_{\mathbf{drt}}$–ID′)
For all closed ACP^-_{drt}-ID terms s and t we have ACP^-_{drt}-ID $\vdash s = t$ if and only if ACP^-_{drt}-ID′ $\vdash s = t$.

Proof It suffices to show that, for closed terms, every axiom of $\mathrm{ACP}^-_{\mathrm{drt}}$–ID′ is derivable from the axioms of $\mathrm{ACP}^-_{\mathrm{drt}}$–ID, and vice versa, that, for closed terms, every axiom of $\mathrm{ACP}^-_{\mathrm{drt}}$–ID is derivable from the axioms of $\mathrm{ACP}^-_{\mathrm{drt}}$–ID′. We can restrict ourselves to the axioms that are not shared by both theories. Furthermore, the proofs regarding Axioms DRTM5–DRTM11 that are given in Theorem 3.3.12 on page 79, with respect to $\mathrm{PA}^-_{\mathrm{drt}}$–ID and $\mathrm{PA}^-_{\mathrm{drt}}$–ID′, remain valid with respect to $\mathrm{ACP}^-_{\mathrm{drt}}$–ID and $\mathrm{ACP}^-_{\mathrm{drt}}$–ID′.

Part I

First, we show that Axioms DRTCM8–DRTCM11 of $\mathrm{ACP}^-_{\mathrm{drt}}$–ID′ are derivable in $\mathrm{ACP}^-_{\mathrm{drt}}$–ID:

Axiom DRTCM8

$$\mathrm{ACP}^-_{\mathrm{drt}}\text{–ID} \vdash \underline{\underline{a}} \mid \sigma_{\mathrm{rel}}(x) = \underline{\underline{\delta}}$$

Consider the following derivation:

$$\mathrm{ACP}^-_{\mathrm{drt}}\text{–ID} \vdash \underline{\underline{a}} \mid \sigma_{\mathrm{rel}}(x) = \nu_{\mathrm{rel}}(\underline{\underline{a}}) \mid \sigma_{\mathrm{rel}}(x) = \underline{\underline{\delta}}$$

Axiom DRTCM9

$$\mathrm{ACP}^-_{\mathrm{drt}}\text{–ID} \vdash \sigma_{\mathrm{rel}}(x) \mid \underline{\underline{a}} = \underline{\underline{\delta}}$$

This axiom is treated symmetrically to the previous axiom.

Axiom DRTCM10

$$\mathrm{ACP}^-_{\mathrm{drt}}\text{–ID} \vdash \underline{\underline{a}} \cdot x \mid \sigma_{\mathrm{rel}}(y) = \underline{\underline{\delta}}$$

Consider the following derivation:

$$\mathrm{ACP}^-_{\mathrm{drt}}\text{–ID} \vdash \underline{\underline{a}} \cdot x \mid \sigma_{\mathrm{rel}}(y) = \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot x \mid \sigma_{\mathrm{rel}}(y) = \nu_{\mathrm{rel}}(\underline{\underline{a}} \cdot x) \mid \sigma_{\mathrm{rel}}(y) = \underline{\underline{\delta}}$$

Axiom DRTCM11

$$\mathrm{ACP}^-_{\mathrm{drt}}\text{–ID} \vdash \sigma_{\mathrm{rel}}(x) \mid \underline{\underline{a}} \cdot y = \underline{\underline{\delta}}$$

This axiom is treated symmetrically to the previous axiom.

Part II

Secondly, we show that Axioms DRTCM6–DRTCM7 of $\mathrm{ACP}^-_{\mathrm{drt}}$–ID are also derivable in $\mathrm{ACP}^-_{\mathrm{drt}}$–ID′:

Axiom DRTCM6

$$\mathrm{ACP}^-_{\mathrm{drt}}\text{–ID}' \vdash \sigma_{\mathrm{rel}}(x) \mid \nu_{\mathrm{rel}}(y) = \underline{\underline{\delta}}$$

Use the general form of basic term y. Take:

$$y \equiv \sum_{i<m} \underline{\underline{a_i}} \cdot t_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \sigma_{\mathrm{rel}}(u_k)$$

for $m, n, p \in \mathbb{N}$, $a_i, b_j \in A_\delta$, and basic terms t_i and u_k. Then we have:

$$\begin{aligned}
\mathrm{ACP}^-_{\mathrm{drt}}\text{–ID}' \vdash\ & \sigma_{\mathrm{rel}}(x) \mid \nu_{\mathrm{rel}}(y) = \\
& \sigma_{\mathrm{rel}}(x) \mid \nu_{\mathrm{rel}}\left(\sum_{i<m} \underline{\underline{a_i}} \cdot t_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \sigma_{\mathrm{rel}}(u_k)\right) = \\
& \sigma_{\mathrm{rel}}(x) \mid \left(\sum_{i<m} \nu_{\mathrm{rel}}(\underline{\underline{a_i}} \cdot t_i) + \sum_{j<n} \nu_{\mathrm{rel}}(\underline{\underline{b_j}}) + \sum_{k<p} \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(u_k))\right) = \\
& \sigma_{\mathrm{rel}}(x) \mid \left(\sum_{i<m} \nu_{\mathrm{rel}}(\underline{\underline{a_i}}) \cdot t_i + \sum_{j<n} \nu_{\mathrm{rel}}(\underline{\underline{b_j}}) + \sum_{k<p} \nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(u_k))\right) =
\end{aligned}$$

$$
\begin{aligned}
&\sigma_{\mathrm{rel}}(x) \mid \left(\sum_{i<m} \underline{\underline{a_i}} \cdot t_i + \sum_{j<n} \underline{\underline{b_j}} + \sum_{k<p} \underline{\underline{\delta}}\right) = \\
&\left(\sigma_{\mathrm{rel}}(x) \mid \sum_{i<m} \underline{\underline{a_i}} \cdot t_i\right) + \left(\sigma_{\mathrm{rel}}(x) \mid \sum_{j<n} \underline{\underline{b_j}}\right) + \left(\sigma_{\mathrm{rel}}(x) \mid \sum_{k<p} \underline{\underline{\delta}}\right) = \\
&\left(\sum_{i<m} \sigma_{\mathrm{rel}}(x) \mid \underline{\underline{a_i}} \cdot t_i\right) + \left(\sum_{j<n} \sigma_{\mathrm{rel}}(x) \mid \underline{\underline{b_j}}\right) + \left(\sum_{k<p} \sigma_{\mathrm{rel}}(x) \mid \underline{\underline{\delta}}\right) = \\
&\sum_{i<m} \underline{\underline{\delta}} + \sum_{j<n} \underline{\underline{\delta}} + \sum_{k<p} \underline{\underline{\delta}} = \\
&\underline{\underline{\delta}}
\end{aligned}
$$

Axiom DRTCM7

$$\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–}\mathrm{ID}' \vdash \nu_{\mathrm{rel}}(x) \mid \sigma_{\mathrm{rel}}(y) = \underline{\underline{\delta}}$$

This axiom is treated symmetrically to the previous axiom.

■

Theorem 3.5.11 (Ground Equivalence of $T(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–}\mathrm{ID})$ and $T(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–}\mathrm{ID}'))$
The term deduction systems $T(ACP^{-}_{drt}\text{–}ID)$ and $T(ACP^{-}_{drt}\text{–}ID')$ are ground equivalent.

Proof Both term deduction systems have the same signature and the same set of deduction rules. Then it is trivial that the same equalities hold between closed terms. ■

Corollary 3.5.12 (Soundness of $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–}\mathrm{ID}'$)
The set of all closed $ACP^{-}_{drt}\text{–}ID'$ terms modulo bisimulation equivalence is a model of $ACP^{-}_{drt}\text{–}ID'$.

Proof This follows immediately from the following observations and Theorem 3.3.15:

(i). $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID and $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′ are ground equivalent equational specifications (see Theorem 3.5.10),

(ii). $T(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–}\mathrm{ID})$ and $T(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–}\mathrm{ID}')$ are ground equivalent term deduction systems (see Theorem 3.5.11),

(iii). Soundness of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID (see Theorem 3.4.12).

■

Remark 3.5.13 (Soundness of $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–}\mathrm{ID}'$)
Soundness of a slightly different version of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′ is also claimed (without proof) in Theorem 3.6.5 of [13] (where $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′ is called $\mathrm{ACP}_{\mathrm{dt}}$).

Corollary 3.5.14 (Completeness of $\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–}\mathrm{ID}'$)
The equational specification $ACP^{-}_{drt}\text{–}ID'$ is a complete axiomatization of the set of closed $ACP^{-}_{drt}\text{–}ID'$ terms modulo bisimulation equivalence.

Proof This follows immediately from the following observations and Theorem 3.3.15:

(i). $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID and $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′ are ground equivalent equational specifications (see Theorem 3.5.10),

(ii). $T(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID})$ and $T(\mathrm{ACP}^{-}_{\mathrm{drt}}\text{–ID}')$ are ground equivalent term deduction systems (see Theorem 3.5.11),

(iii). Completeness of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID (see Theorem 3.4.15).

■

Remark 3.5.15 (Completeness of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′)
Completeness of a slightly different version of $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′ is also claimed (without proof) in Theorem 3.6.7 of [13] (where $\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID′ is called $\mathrm{ACP}_{\mathrm{dt}}$).

3.6 Soundness and Completeness of $\mathrm{PA}^{+}_{\mathrm{drt}}$

Definition 3.6.1 (Signature of $\mathrm{PA}_{\mathrm{drt}}$)
The signature of $\mathrm{PA}_{\mathrm{drt}}$ consists of the *undelayable atomic actions* $\{\underline{\underline{a}} | a \in A\}$, the *delayable atomic actions* $\{\underline{a} | a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *delayable deadlock constant* δ, the *immediate deadlock constant* $\dot{\delta}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *time unit delay operator* σ_{rel}, the *"now" operator* ν_{rel}, the *unbounded start delay operator* $\lfloor\ \rfloor^{\omega}$, the *(free) merge operator* $\|$, and the *left merge operator* $⫿\!\!\!\!\_$.

Definition 3.6.2 (Axioms of $\mathrm{PA}_{\mathrm{drt}}$)
The process algebra $\mathrm{PA}_{\mathrm{drt}}$ is axiomatized by the axioms of $\mathrm{BPA}_{\mathrm{drt}}$ given in Definition 2.8.2 on page 54, Axioms DRTM1 and DRTM4 shown in Table 24 on page 69, Axiom DRTM6 shown in Table 25 on page 70, and finally Axioms DRTM2ID–DRTM3ID, DRTM5ID, and DRTMID1–DRTMID2 shown in Table 34: $\mathrm{PA}_{\mathrm{drt}}$ = A1–A5 + A6ID + A7ID + DRT1–DRT5 + DRTSID + DCS1–DCS4 + DCSID + ATS + USD + DRTM1 + DRTM2ID–DRTM3ID + DRTM4 + DRTM5ID + DRTM6 + DRTMID1–DRTMID2.

$\underline{\underline{a}} \mathbin{⫿\!\!\_} (x + \underline{\underline{\delta}}) = \underline{\underline{a}} \cdot (x + \underline{\underline{\delta}})$	DRTM2ID
$\underline{\underline{a}} \cdot x \mathbin{⫿\!\!\_} (y + \underline{\underline{\delta}}) = \underline{\underline{a}} \cdot (x \,\|\, (y + \underline{\underline{\delta}}))$	DRTM3ID
$\sigma_{\mathrm{rel}}(x) \mathbin{⫿\!\!\_} (\nu_{\mathrm{rel}}(y) + \underline{\underline{\delta}}) = \underline{\underline{\delta}}$	DRTM5ID
$x \mathbin{⫿\!\!\_} \dot{\delta} = \dot{\delta}$	DRTMID1
$\dot{\delta} \mathbin{⫿\!\!\_} x = \dot{\delta}$	DRTMID2

Table 34: Additional axioms for $\mathrm{PA}_{\mathrm{drt}}$.

Definition 3.6.3 (Semantics of $\mathrm{PA}_{\mathrm{drt}}$)
The semantics of $\mathrm{PA}_{\mathrm{drt}}$ are given by the term deduction system $T(\mathrm{PA}_{\mathrm{drt}})$ induced by the deduction rules for $\mathrm{BPA}_{\mathrm{drt}}$ given in Definition 2.8.4 on page 54 and the deduction rules for the (free) merge with immediate deadlock shown in Table 35 on the next page.

$\dfrac{x \xrightarrow{a} x',\ \neg\mathrm{ID}(y)}{x \parallel y \xrightarrow{a} x' \parallel y}$	$\dfrac{y \xrightarrow{a} y',\ \neg\mathrm{ID}(x)}{x \parallel y \xrightarrow{a} x \parallel y'}$	$\dfrac{x \xrightarrow{a} x',\ \neg\mathrm{ID}(y)}{x \mathbin{\|\!\!\lfloor} y \xrightarrow{a} x' \parallel y}$
$\dfrac{x \xrightarrow{a} \surd,\ \neg\mathrm{ID}(y)}{x \parallel y \xrightarrow{a} y}$	$\dfrac{y \xrightarrow{a} \surd,\ \neg\mathrm{ID}(x)}{x \parallel y \xrightarrow{a} x}$	$\dfrac{x \xrightarrow{a} \surd,\ \neg\mathrm{ID}(y)}{x \mathbin{\|\!\!\lfloor} y \xrightarrow{a} y}$
$\dfrac{x \xrightarrow{\sigma} x',\ y \xrightarrow{\sigma} y'}{x \parallel y \xrightarrow{\sigma} x' \parallel y'}$	$\dfrac{x \xrightarrow{\sigma} x',\ y \xrightarrow{\sigma} y'}{x \mathbin{\|\!\!\lfloor} y \xrightarrow{\sigma} x' \mathbin{\|\!\!\lfloor} y'}$	
$\dfrac{\mathrm{ID}(x)}{\mathrm{ID}(x \parallel y)}$	$\dfrac{\mathrm{ID}(y)}{\mathrm{ID}(x \parallel y)}$	
$\dfrac{\mathrm{ID}(x)}{\mathrm{ID}(x \mathbin{\|\!\!\lfloor} y)}$	$\dfrac{\mathrm{ID}(y)}{\mathrm{ID}(x \mathbin{\|\!\!\lfloor} y)}$	

Table 35: Deduction rules for the (free) merge with immediate deadlock.

Definition 3.6.4 (Bisimulation and Bisimulation Model for $\mathrm{PA_{drt}}$)
Bisimulation for $\mathrm{PA_{drt}}$ and the corresponding bisimulation model are defined in the same way as for $\mathrm{BPA^{-}_{drt}}$ and BPA respectively. Replace "$\mathrm{BPA^{-}_{drt}}$" by "$\mathrm{PA_{drt}}$" in Definition 2.7.7 on page 46 and "BPA" by "$\mathrm{PA_{drt}}$" in Definition 2.2.11 on page 8.

Definition 3.6.5 (Basic Terms of $\mathrm{PA_{drt}}$)
If we speak of basic terms in the context of $\mathrm{PA_{drt}}$, we mean $(\sigma, \underline{\underline{\delta}}, \delta, \dot{\delta})$-basic terms as defined in Definition 2.8.7 on page 54.

Definition 3.6.6 (Number of Symbols of a $\mathrm{PA_{drt}}$ term)
We define $n(x)$, the number of symbols of x, inductively as follows:

(i). We define $n(\dot{\delta}) = 1$,

(ii). for $a \in A_\delta$, we define $n(\underline{\underline{a}}) = n(a) = 1$,

(iii). for closed $\mathrm{PA_{drt}}$ terms x and y, we define $n(x+y) = n(x \cdot y) = n(x \parallel y) = n(x \mathbin{\|\!\!\lfloor} y) = n(x) + n(y) + 1$,

(iv). for a closed $\mathrm{PA_{drt}}$ term x, we define $n(\sigma_{\mathrm{rel}}(x)) = n(\nu_{\mathrm{rel}}(x)) = n(\lfloor x \rfloor^{\omega}) = n(x) + 1$.

Proposition 3.6.7 (Properties of $\mathrm{PA^{+}_{drt}}$)
For PA_{drt} terms x and y, and any $a \in A_\delta$, we have the following equalities:

(i). $PA^{+}_{drt} \vdash \lfloor a \rfloor^{\omega} = a$

(ii). $PA^{+}_{drt} \vdash \lfloor x \cdot y \rfloor^{\omega} = \lfloor x \rfloor^{\omega} \cdot y$

(iii). $PA^{+}_{drt} \vdash \lfloor x + y \rfloor^{\omega} = \lfloor x \rfloor^{\omega} + \lfloor y \rfloor^{\omega}$

(iv). $PA^{+}_{drt} \vdash \lfloor \sigma_{rel}(x) \rfloor^{\omega} = \delta$

(v). $PA^{+}_{drt} \vdash \lfloor \dot{\delta} \rfloor^{\omega} = \delta$

(vi). $PA^{+}_{drt} \vdash a \mathbin{⫫} \lfloor x \rfloor^{\omega} = a \cdot \lfloor x \rfloor^{\omega}$

(vii). $PA^{+}_{drt} \vdash a \cdot x \mathbin{⫫} \lfloor y \rfloor^{\omega} = a \cdot (x \parallel \lfloor y \rfloor^{\omega})$

(viii). $PA_{drt} \vdash \nu_{rel}(a) = \underline{\underline{a}}$

(ix). $PA_{drt} \vdash \lfloor x \rfloor^{\omega} + \underline{\underline{\delta}} = \lfloor x \rfloor^{\omega}$

Proof The proofs for equality (i)–(v) and (viii)–(ix) given in Proposition 2.8.11 on page 55, with respect to $\mathrm{BPA_{drt}}$, remain valid in the setting of $\mathrm{PA_{drt}}$, as can be easily checked.

Equality (vi) and (vii) do not appear in Proposition 2.8.11. Consider the following computation for equality (vi):

$$\begin{aligned}
\mathrm{PA_{drt}} \vdash a \mathbin{⫫} \lfloor x \rfloor^{\omega} &= \lfloor \underline{\underline{a}} \rfloor^{\omega} \mathbin{⫫} \lfloor x \rfloor^{\omega} \\
&= (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega})) \mathbin{⫫} \lfloor x \rfloor^{\omega} \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \mathbin{⫫} \lfloor x \rfloor^{\omega} \\
&= \underline{\underline{a}} \mathbin{⫫} \lfloor x \rfloor^{\omega} + \sigma_{\mathrm{rel}}(a) \mathbin{⫫} \lfloor x \rfloor^{\omega} \\
&= \underline{\underline{a}} \mathbin{⫫} (\lfloor x \rfloor^{\omega} + \underline{\underline{\delta}}) + \sigma_{\mathrm{rel}}(a) \mathbin{⫫} (\nu_{\mathrm{rel}}(x) + \sigma_{\mathrm{rel}}(\lfloor x \rfloor^{\omega})) \\
&= \underline{\underline{a}} \cdot (\lfloor x \rfloor^{\omega} + \underline{\underline{\delta}}) + \sigma_{\mathrm{rel}}(a \mathbin{⫫} \lfloor x \rfloor^{\omega}) \\
&= \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot \lfloor x \rfloor^{\omega} + \sigma_{\mathrm{rel}}(a \mathbin{⫫} \lfloor x \rfloor^{\omega}) \\
&= \nu_{\mathrm{rel}}(\underline{\underline{a}} \cdot \lfloor x \rfloor^{\omega}) + \sigma_{\mathrm{rel}}(a \mathbin{⫫} \lfloor x \rfloor^{\omega})
\end{aligned}$$

Using RSP(USD) we obtain:

$$\mathrm{PA^{+}_{drt}} \vdash a \mathbin{⫫} \lfloor x \rfloor^{\omega} = \lfloor \underline{\underline{a}} \cdot \lfloor x \rfloor^{\omega} \rfloor^{\omega} = \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot \lfloor x \rfloor^{\omega} = a \cdot \lfloor x \rfloor^{\omega}$$

Finally, consider the following computation for equality (vii):

$$\begin{aligned}
\mathrm{PA_{drt}} \vdash a \cdot x \mathbin{⫫} \lfloor y \rfloor^{\omega} &= \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot x \mathbin{⫫} \lfloor y \rfloor^{\omega} \\
&= (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega})) \cdot x \mathbin{⫫} \lfloor y \rfloor^{\omega} \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \cdot x \mathbin{⫫} \lfloor y \rfloor^{\omega} \\
&= (\underline{\underline{a}} \cdot x + \sigma_{\mathrm{rel}}(a \cdot x)) \mathbin{⫫} \lfloor y \rfloor^{\omega} \\
&= \underline{\underline{a}} \cdot x \mathbin{⫫} \lfloor y \rfloor^{\omega} + \sigma_{\mathrm{rel}}(a \cdot x) \mathbin{⫫} \lfloor y \rfloor^{\omega} \\
&= \underline{\underline{a}} \cdot x \mathbin{⫫} (\lfloor y \rfloor^{\omega} + \underline{\underline{\delta}}) + \sigma_{\mathrm{rel}}(a \cdot x) \mathbin{⫫} (\nu_{\mathrm{rel}}(y) + \sigma_{\mathrm{rel}}(\lfloor y \rfloor^{\omega})) \\
&= \underline{\underline{a}} \cdot (x \parallel (\lfloor y \rfloor^{\omega} + \underline{\underline{\delta}})) + \sigma_{\mathrm{rel}}(a \cdot x \mathbin{⫫} \lfloor y \rfloor^{\omega}) \\
&= \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot (x \parallel \lfloor y \rfloor^{\omega}) + \sigma_{\mathrm{rel}}(a \cdot x \mathbin{⫫} \lfloor y \rfloor^{\omega}) \\
&= \nu_{\mathrm{rel}}(\underline{\underline{a}} \cdot (x \parallel \lfloor y \rfloor^{\omega})) + \sigma_{\mathrm{rel}}(a \cdot x \mathbin{⫫} \lfloor y \rfloor^{\omega})
\end{aligned}$$

Using RSP(USD) we obtain:

$$\mathrm{PA^{+}_{drt}} \vdash a \cdot x \mathbin{⫫} \lfloor y \rfloor^{\omega} = \lfloor \underline{\underline{a}} \cdot (x \parallel \lfloor y \rfloor^{\omega}) \rfloor^{\omega} = \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot (x \parallel \lfloor y \rfloor^{\omega}) = a \cdot (x \parallel \lfloor y \rfloor^{\omega})$$

■

Theorem 3.6.8 (Elimination for $\mathbf{PA^{+}_{drt}}$)
Let t be a closed PA_{drt} term. Then there is a closed BPA_{drt} term s such that $PA^{+}_{drt} \vdash t = s$.

Proof Let t be a closed PA_{drt} term. The theorem is proven by induction on $n(t)$ and case distinction on the general structure of t.

(i). $t \equiv \dot{\delta}$. Then t is a closed BPA_{drt} term.

(ii). $t \equiv \underline{\underline{a}}$ for some $a \in A_\delta$. Then t is a closed BPA_{drt} term.

(iii). $t \equiv a$ for some $a \in A_\delta$. Then t is a closed BPA_{drt} term.

(iv). $t \equiv t_1 + t_2$ for closed PA_{drt} terms t_1 and t_2. By induction there are closed BPA_{drt} terms s_1 and s_2 such that $\text{PA}^+_{\text{drt}} \vdash t_1 = s_1$ and $\text{PA}^+_{\text{drt}} \vdash t_2 = s_2$. But then also $\text{PA}^+_{\text{drt}} \vdash t_1 + t_2 = s_1 + s_2$ and $s_1 + s_2$ is a closed BPA_{drt} term.

(v). $t \equiv t_1 \cdot t_2$ for closed PA_{drt} terms t_1 and t_2. This case is treated analogously to case (ii).

(vi). $t \equiv \sigma_{\text{rel}}(t_1)$ for a closed PA_{drt} term t_1. This case is treated analogously to case (ii).

(vii). $t \equiv \nu_{\text{rel}}(t_1)$ for a closed PA_{drt} term t_1. This case is treated analogously to case (ii).

(viii). $t \equiv \lfloor t_1 \rfloor^\omega$ for a closed PA_{drt} term t_1. This case is treated analogously to case (ii).

(ix). $t \equiv t_1 \mathbin{\lfloor\!\!\lfloor} t_2$ for closed PA_{drt} terms t_1 and t_2. By induction there are closed BPA_{drt} terms s_1 and s_2 such that $\text{PA}^+_{\text{drt}} \vdash t_1 = s_1$ and $\text{PA}^+_{\text{drt}} \vdash t_2 = s_2$. By Theorem 2.8.16, the elimination theorem for BPA_{drt}, there are basic terms r_1 and r_2 such that $\text{BPA}^+_{\text{drt}} \vdash s_1 = r_1$ and $\text{BPA}^+_{\text{drt}} \vdash s_2 = r_2$. But then also, $\text{PA}^+_{\text{drt}} \vdash t_1 = r_1$, $\text{PA}^+_{\text{drt}} \vdash t_2 = r_2$, and $\text{PA}^+_{\text{drt}} \vdash t_1 \mathbin{\lfloor\!\!\lfloor} t_2 = r_1 \mathbin{\lfloor\!\!\lfloor} r_2$. We proceed by induction on the structure of basic terms, and distinguish all possible cases for basic term r_1:

(a) $r_1 \equiv \dot{\delta}$. Then $\text{PA}^+_{\text{drt}} \vdash t_1 \mathbin{\lfloor\!\!\lfloor} t_2 = r_1 \mathbin{\lfloor\!\!\lfloor} r_2 = \dot{\delta} \mathbin{\lfloor\!\!\lfloor} r_2 = \dot{\delta}$, and $\dot{\delta}$ is a closed BPA_{drt} term.

(b) $r_1 \equiv \underline{\underline{a}}$ for some $a \in A_\delta$. Using Lemma 2.8.14 we distinguish two cases:

i. $r_2 = \dot{\delta}$. Then we have: $\text{PA}^+_{\text{drt}} \vdash t_1 \mathbin{\lfloor\!\!\lfloor} t_2 = r_1 \mathbin{\lfloor\!\!\lfloor} r_2 = r_1 \mathbin{\lfloor\!\!\lfloor} \dot{\delta} = \dot{\delta}$, and $\dot{\delta}$ is a closed BPA_{drt} term.

ii. $r_2 = r_2 + \underline{\underline{\delta}}$. Then we have: $\text{PA}^+_{\text{drt}} \vdash t_1 \mathbin{\lfloor\!\!\lfloor} t_2 = r_1 \mathbin{\lfloor\!\!\lfloor} r_2 = \underline{\underline{a}} \mathbin{\lfloor\!\!\lfloor} (r_2 + \underline{\underline{\delta}}) = \underline{\underline{a}} \cdot (r_2 + \underline{\underline{\delta}}) = \underline{\underline{a}} \cdot r_2$, and $\underline{\underline{a}} \cdot r_2$ is a closed BPA_{drt} term.

(c) $r_1 \equiv a$ for some $a \in A_\delta$. Using Lemma 2.8.13 we distinguish four cases:

i. $r_2 = \dot{\delta}$. Then we have: $\text{PA}^+_{\text{drt}} \vdash t_1 \mathbin{\lfloor\!\!\lfloor} t_2 = r_1 \mathbin{\lfloor\!\!\lfloor} r_2 = r_1 \mathbin{\lfloor\!\!\lfloor} \dot{\delta} = \dot{\delta}$, and $\dot{\delta}$ is a closed BPA_{drt} term.

ii. $r_2 = \nu_{\text{rel}}(r_2) + \underline{\underline{\delta}}$. Then we have:

$$\begin{aligned}
\text{PA}^+_{\text{drt}} \vdash t_1 \mathbin{\lfloor\!\!\lfloor} t_2 &= r_1 \mathbin{\lfloor\!\!\lfloor} r_2 \\
&= a \mathbin{\lfloor\!\!\lfloor} r_2 \\
&= \lfloor \underline{\underline{a}} \rfloor^\omega \mathbin{\lfloor\!\!\lfloor} r_2 \\
&= (\nu_{\text{rel}}(\underline{\underline{a}}) + \sigma_{\text{rel}}(\lfloor \underline{\underline{a}} \rfloor^\omega)) \mathbin{\lfloor\!\!\lfloor} r_2 \\
&= (\underline{\underline{a}} + \sigma_{\text{rel}}(a)) \mathbin{\lfloor\!\!\lfloor} r_2 \\
&= \underline{\underline{a}} \mathbin{\lfloor\!\!\lfloor} r_2 + \sigma_{\text{rel}}(a) \mathbin{\lfloor\!\!\lfloor} r_2 \\
&= \underline{\underline{a}} \mathbin{\lfloor\!\!\lfloor} (\nu_{\text{rel}}(r_2) + \underline{\underline{\delta}}) + \sigma_{\text{rel}}(a) \mathbin{\lfloor\!\!\lfloor} (\nu_{\text{rel}}(r_2) + \underline{\underline{\delta}})
\end{aligned}$$

$$
\begin{aligned}
&= \underline{\underline{a}} \cdot (\nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}) + \underline{\underline{\delta}} \\
&= \underline{\underline{a}} \cdot r_2 + \underline{\underline{\delta}} \\
&= \underline{\underline{a}} \cdot r_2 + \underline{\underline{\delta}} \cdot r_2 \\
&= (\underline{\underline{a}} + \underline{\underline{\delta}}) \cdot r_2 \\
&= \underline{\underline{a}} \cdot r_2,
\end{aligned}
$$

and $\underline{\underline{a}} \cdot r_2$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

iii. $r_2 = \lfloor r_2 \rfloor^{\omega}$. Then, using Proposition 3.6.7(vi), we have: $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\lfloor} t_2 = r_1 \mathbin{\|\!\!\lfloor} r_2 = a \mathbin{\|\!\!\lfloor} \lfloor r_2 \rfloor^{\omega} = a \cdot \lfloor r_2 \rfloor^{\omega} = a \cdot r_2$, and $a \cdot r_2$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

iv. $r_2 = \nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')$ for a basic term r_2' such that $n(r_2') < n(r_2)$. Then we have:

$$
\begin{aligned}
\mathrm{PA}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\lfloor} t_2 &= r_1 \mathbin{\|\!\!\lfloor} r_2 \\
&= a \mathbin{\|\!\!\lfloor} r_2 \\
&= \lfloor \underline{a} \rfloor^{\omega} \mathbin{\|\!\!\lfloor} r_2 \\
&= (\nu_{\mathrm{rel}}(\underline{a}) + \sigma_{\mathrm{rel}}(\lfloor \underline{a} \rfloor^{\omega})) \mathbin{\|\!\!\lfloor} r_2 \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \mathbin{\|\!\!\lfloor} r_2 \\
&= \underline{\underline{a}} \mathbin{\|\!\!\lfloor} r_2 + \sigma_{\mathrm{rel}}(a) \mathbin{\|\!\!\lfloor} r_2 \\
&= \underline{\underline{a}} \mathbin{\|\!\!\lfloor} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) + \\
&\qquad \sigma_{\mathrm{rel}}(a) \mathbin{\|\!\!\lfloor} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) \\
&= \underline{\underline{a}} \mathbin{\|\!\!\lfloor} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2') + \underline{\underline{\delta}}) + \sigma_{\mathrm{rel}}(a \mathbin{\|\!\!\lfloor} r_2') \\
&= \underline{\underline{a}} \cdot (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2') + \underline{\underline{\delta}}) + \sigma_{\mathrm{rel}}(a \mathbin{\|\!\!\lfloor} r_2') \\
&= \underline{\underline{a}} \cdot (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) + \sigma_{\mathrm{rel}}(a \mathbin{\|\!\!\lfloor} r_2') \\
&= \underline{\underline{a}} \cdot r_2 + \sigma_{\mathrm{rel}}(a \mathbin{\|\!\!\lfloor} r_2').
\end{aligned}
$$

By the induction hypothesis there exists a closed $\mathrm{BPA}_{\mathrm{drt}}$ term p such that $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash a \mathbin{\|\!\!\lfloor} r_2' = p$. Then, $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\lfloor} t_2 = \underline{\underline{a}} \cdot r_2 + \sigma_{\mathrm{rel}}(a \mathbin{\|\!\!\lfloor} r_2') = \underline{\underline{a}} \cdot r_2 + \sigma_{\mathrm{rel}}(p)$, and $\underline{\underline{a}} \cdot r_2 + \sigma_{\mathrm{rel}}(p)$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(d) $r_1 \equiv \underline{\underline{a}} \cdot r_1'$ for some $a \in A_{\delta}$ and basic term r_1'. Using Lemma 2.8.14 we distinguish two cases:

i. $r_2 = \dot{\delta}$. Then we have: $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\lfloor} t_2 = r_1 \mathbin{\|\!\!\lfloor} r_2 = r_1 \mathbin{\|\!\!\lfloor} \dot{\delta} = \dot{\delta}$, and $\dot{\delta}$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

ii. $r_2 = r_2 + \underline{\underline{\delta}}$. Then we have: $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\lfloor} t_2 = r_1 \mathbin{\|\!\!\lfloor} r_2 = \underline{\underline{a}} \cdot r_1' \mathbin{\|\!\!\lfloor} (r_2 + \underline{\underline{\delta}}) = \underline{\underline{a}} \cdot (r_1' \parallel (r_2 + \underline{\underline{\delta}})) = \underline{\underline{a}} \cdot (r_1' \parallel r_2)$. By the induction hypothesis there exists a closed $\mathrm{BPA}_{\mathrm{drt}}$ term p such that $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash r_1' \parallel r_2 = p$. Then, $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\lfloor} t_2 = \underline{\underline{a}} \cdot (r_1' \parallel r_2) = \underline{\underline{a}} \cdot p$, and $\underline{\underline{a}} \cdot p$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(e) $r_1 \equiv a \cdot r_1'$ for some $a \in A_{\delta}$ and basic term r_1'. Using Lemma 2.8.13 we distinguish four cases:

i. $r_2 = \dot{\delta}$. Then we have: $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\lfloor} t_2 = r_1 \mathbin{\|\!\!\lfloor} r_2 = r_1 \mathbin{\|\!\!\lfloor} \dot{\delta} = \dot{\delta}$, and $\dot{\delta}$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

ii. $r_2 = \nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}$. Then we have:

$$
\begin{aligned}
\mathrm{PA}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\lfloor} t_2 &= r_1 \mathbin{\|\!\!\lfloor} r_2 \\
&= a \cdot r_1' \mathbin{\|\!\!\lfloor} r_2
\end{aligned}
$$

$$
\begin{aligned}
&= \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega})) \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= (\underline{\underline{a}} \cdot r_1' + \sigma_{\mathrm{rel}}(a) \cdot r_1') \mathbin{\|\!\_} r_2 \\
&= (\underline{\underline{a}} \cdot r_1' + \sigma_{\mathrm{rel}}(a \cdot r_1')) \mathbin{\|\!\_} r_2 \\
&= \underline{\underline{a}} \cdot r_1' \mathbin{\|\!\_} r_2 + \sigma_{\mathrm{rel}}(a \cdot r_1') \mathbin{\|\!\_} r_2 \\
&= \underline{\underline{a}} \cdot r_1' \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}) + \sigma_{\mathrm{rel}}(a \cdot r_1') \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}) \\
&= \underline{\underline{a}} \cdot (r_1' \parallel (\nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}})) + \underline{\underline{\delta}} \\
&= \underline{\underline{a}} \cdot (r_1' \parallel r_2) + \underline{\underline{\delta}} \\
&= \underline{\underline{a}} \cdot (r_1' \parallel r_2) + \underline{\underline{\delta}} \cdot (r_1' \parallel r_2) \\
&= (\underline{\underline{a}} + \underline{\underline{\delta}}) \cdot (r_1' \parallel r_2) \\
&= \underline{\underline{a}} \cdot (r_1' \parallel r_2).
\end{aligned}
$$

By the induction hypothesis there exists a closed $\mathrm{BPA}_{\mathrm{drt}}$ term p such that $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash r_1' \parallel r_2 = p$. Then, $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = \underline{\underline{a}} \cdot (r_1' \parallel r_2) = \underline{\underline{a}} \cdot p$, and $\underline{\underline{a}} \cdot p$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

iii. $r_2 = \lfloor r_2 \rfloor^{\omega}$. Then, using Proposition 3.6.7(vii), we have: $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1 \mathbin{\|\!\_} r_2 = a \cdot r_1' \mathbin{\|\!\_} \lfloor r_2 \rfloor^{\omega} = a \cdot (r_1' \parallel \lfloor r_2 \rfloor^{\omega}) = a \cdot (r_1' \parallel r_2)$. By the induction hypothesis there exists a closed $\mathrm{BPA}_{\mathrm{drt}}$ term p such that $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash r_1' \parallel r_2 = p$. Then, $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = a \cdot (r_1' \parallel r_2) = a \cdot p$, and $a \cdot p$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

iv. $r_2 = \nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')$ for a basic term r_2' such that $n(r_2') < n(r_2)$. Then we have:

$$
\begin{aligned}
\mathrm{PA}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 &= r_1 \mathbin{\|\!\_} r_2 \\
&= a \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega})) \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= (\underline{\underline{a}} \cdot r_1' + \sigma_{\mathrm{rel}}(a) \cdot r_1') \mathbin{\|\!\_} r_2 \\
&= (\underline{\underline{a}} \cdot r_1' + \sigma_{\mathrm{rel}}(a \cdot r_1')) \mathbin{\|\!\_} r_2 \\
&= \underline{\underline{a}} \cdot r_1' \mathbin{\|\!\_} r_2 + \sigma_{\mathrm{rel}}(a \cdot r_1') \mathbin{\|\!\_} r_2 \\
&= \underline{\underline{a}} \cdot r_1' \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) + {} \\
&\qquad \sigma_{\mathrm{rel}}(a \cdot r_1') \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) \\
&= \underline{\underline{a}} \cdot r_1' \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2') + \underline{\underline{\delta}}) + {} \\
&\qquad \sigma_{\mathrm{rel}}(a \cdot r_1') \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) \\
&= \underline{\underline{a}} \cdot (r_1' \parallel (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2') + \underline{\underline{\delta}})) + \sigma_{\mathrm{rel}}(a \cdot r_1' \mathbin{\|\!\_} r_2') \\
&= \underline{\underline{a}} \cdot (r_1' \parallel (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2'))) + \sigma_{\mathrm{rel}}(r_1 \mathbin{\|\!\_} r_2') \\
&= \underline{\underline{a}} \cdot (r_1' \parallel r_2) + \sigma_{\mathrm{rel}}(r_1 \mathbin{\|\!\_} r_2').
\end{aligned}
$$

By the induction hypothesis there exist closed $\mathrm{BPA}_{\mathrm{drt}}$ terms p_1 and p_2 such that $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash r_1' \parallel r_2 = p_1$ and $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash r_1 \mathbin{\|\!\_} r_2' = p_2$. Then, $\mathrm{PA}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 =$

$\underline{\underline{a}} \cdot (r_1' \parallel r_2) + \sigma_{\mathrm{rel}}(r_1 \mathbin{\|\!\_} r_2') = \underline{\underline{a}} \cdot p_1 + \sigma_{\mathrm{rel}}(p_2)$, and $\underline{\underline{a}} \cdot p_1 + \sigma_{\mathrm{rel}}(p_2)$ is a closed $\mathrm{BPA_{drt}}$ term.

(f) $r_1 \equiv r_1' + r_1''$ for basic terms r_1' and r_1''. Then $\mathrm{PA}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1 \mathbin{\|\!\_} r_2 = (r_1' + r_1'') \mathbin{\|\!\_} r_2 = r_1' \mathbin{\|\!\_} r_2 + r_1'' \mathbin{\|\!\_} r_2$. By induction there exist closed $\mathrm{BPA_{drt}}$ terms p_1 and p_2 such that $\mathrm{PA}^+_{\mathrm{drt}} \vdash r_1' \mathbin{\|\!\_} r_2 = p_1$ and $\mathrm{PA}^+_{\mathrm{drt}} \vdash r_1'' \mathbin{\|\!\_} r_2 = p_2$. Then also $\mathrm{PA}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1' \mathbin{\|\!\_} r_2 + r_1'' \mathbin{\|\!\_} r_2 = p_1 + p_2$, and $p_1 + p_2$ is a closed $\mathrm{BPA_{drt}}$ term.

(g) $r_1 \equiv \sigma_{\mathrm{rel}}(r_1')$ for a basic term r_1'. Using Lemma 2.8.13 we distinguish four cases:

i. $r_2 = \dot{\delta}$. Then we have: $\mathrm{PA}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1 \mathbin{\|\!\_} r_2 = a \mathbin{\|\!\_} \dot{\delta} = \dot{\delta}$, and $\dot{\delta}$ is a closed $\mathrm{BPA_{drt}}$ term.

ii. $r_2 = \nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}$. Then $\mathrm{PA}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1 \mathbin{\|\!\_} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}) = \underline{\underline{\delta}}$, and $\underline{\underline{\delta}}$ is a closed $\mathrm{BPA_{drt}}$ term.

iii. $r_2 = \lfloor r_2 \rfloor^{\omega}$. Then we have: $\mathrm{PA}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1 \mathbin{\|\!\_} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{\|\!\_} \lfloor r_2 \rfloor^{\omega} = \sigma_{\mathrm{rel}}(r_1') \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(\lfloor r_2 \rfloor^{\omega}) = \sigma_{\mathrm{rel}}(r_1' \mathbin{\|\!\_} \lfloor r_2 \rfloor^{\omega}) = \sigma_{\mathrm{rel}}(r_1' \mathbin{\|\!\_} r_2)$. By the induction hypothesis there exists a closed $\mathrm{BPA_{drt}}$ term p such that $\mathrm{PA}^+_{\mathrm{drt}} \vdash r_1' \parallel r_2 = p$. Then, $\mathrm{PA}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = \sigma_{\mathrm{rel}}(r_1' \parallel r_2) = \sigma_{\mathrm{rel}}(p)$, and $\sigma_{\mathrm{rel}}(p)$ is a closed $\mathrm{BPA_{drt}}$ term.

iv. $r_2 = \nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')$ for a basic term r_2' such that $n(r_2') < n(r_2)$. Then we have: $\mathrm{PA}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1 \mathbin{\|\!\_} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) = \sigma_{\mathrm{rel}}(r_1' \mathbin{\|\!\_} r_2')$. By the induction hypothesis there is a closed $\mathrm{BPA_{drt}}$ term p such that $\mathrm{PA}^+_{\mathrm{drt}} \vdash r_1' \mathbin{\|\!\_} r_2' = p$. Then, $\mathrm{PA}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = \sigma_{\mathrm{rel}}(r_1' \mathbin{\|\!\_} r_2') = \sigma_{\mathrm{rel}}(p)$, and $\sigma_{\mathrm{rel}}(p)$ is a closed $\mathrm{BPA_{drt}}$ term.

(x). $t \equiv t_1 \parallel t_2$ for closed $\mathrm{PA_{drt}}$ terms t_1 and t_2. Then $\mathrm{PA}^+_{\mathrm{drt}} \vdash t_1 \parallel t_2 = t_1 \mathbin{\|\!\_} t_2 + t_2 \mathbin{\|\!\_} t_1$. By (ix) there are closed $\mathrm{BPA_{drt}}$ terms p_1 and p_2 such that $\mathrm{PA}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = p_1$ and $\mathrm{PA}^+_{\mathrm{drt}} \vdash t_2 \mathbin{\|\!\_} t_1 = p_2$. But then also $\mathrm{PA}^+_{\mathrm{drt}} \vdash t_1 \parallel t_2 = t_1 \mathbin{\|\!\_} t_2 + t_2 \mathbin{\|\!\_} t_1 = p_1 + p_2$, and $p_1 + p_2$ is a closed $\mathrm{BPA_{drt}}$ term.

■

Corollary 3.6.9 (Elimination for $\mathrm{PA}^+_{\mathrm{drt}}$)
Let t be a closed PA_{drt} term. Then there is a basic term s such that $PA^+_{drt} \vdash s = t$.

Proof This follows immediately from:

(i). The elimination theorem for $\mathrm{PA}^+_{\mathrm{drt}}$ (see Theorem 3.6.8),

(ii). the elimination theorem for $\mathrm{BPA}^+_{\mathrm{drt}}$ (see Theorem 2.8.16),

(iii). the fact that all axioms of $\mathrm{BPA}^+_{\mathrm{drt}}$ are also contained in $\mathrm{PA}^+_{\mathrm{drt}}$.

■

Remark 3.6.10 (Elimination for $\mathrm{PA_{drt}}$)
Elimination for a somewhat different version of $\mathrm{PA_{drt}}$ is also claimed (without proof) in Section 3.9 of [10].

Theorem 3.6.11 (Soundness of $\mathrm{PA}^+_{\mathrm{drt}}$)
The set of closed PA_{drt} terms modulo bisimulation equivalence is a model of PA^+_{drt}.

Proof We only prove soundness for the axioms of $\mathrm{PA}_{\mathrm{drt}}$ that have not been treated in earlier soundness proofs. Note that to extend these proofs to $\mathrm{PA}_{\mathrm{drt}}$, we have to check that the bisimulations given in previous soundness proofs respect the ID predicate (as required by transfer condition (iv.) in Definition 2.7.7 on page 46). However, as the fact that they do can be easily checked, we will not give details.

Axiom DRTM2ID Take the relation:

$$R = \{(s,s), (\underline{\underline{a}} \mathbin{\|\!\lfloor} (s + \underline{\underline{\delta}}), \underline{\underline{a}} \cdot (s + \underline{\underline{\delta}})) \,|\, s \in C(\mathrm{PA}_{\mathrm{drt}})\}$$

We look at the transitions of both sides at the same time. The only transition of the left-hand side is $\underline{\underline{a}} \mathbin{\|\!\lfloor} (s + \underline{\underline{\delta}}) \xrightarrow{a} s + \underline{\underline{\delta}}$, and the only transition of the right-hand side is $\underline{\underline{a}} \cdot (s + \underline{\underline{\delta}}) \xrightarrow{a} s + \underline{\underline{\delta}}$, and note that $(s + \underline{\underline{\delta}}, s + \underline{\underline{\delta}}) \in R$. Finally, neither side satisfies the ID predicate: $\neg\mathrm{ID}(\underline{\underline{a}} \mathbin{\|\!\lfloor} (s + \underline{\underline{\delta}}))$ and $\neg\mathrm{ID}(\underline{\underline{a}} \cdot (s + \underline{\underline{\delta}}))$ (note that $\neg\mathrm{ID}(s + \underline{\underline{\delta}})$ even if $\mathrm{ID}(s)$).

Axiom DRTM3ID Take the relation:

$$R = \{(s,s), (\underline{\underline{a}} \cdot s \mathbin{\|\!\lfloor} (t + \underline{\underline{\delta}}), \underline{\underline{a}} \cdot (s \parallel (t + \underline{\underline{\delta}}))) \,|\, s,t \in C(\mathrm{PA}_{\mathrm{drt}})\}$$

We look at the transitions of both sides at the same time. The only transition of the left-hand side is $\underline{\underline{a}} \cdot s \mathbin{\|\!\lfloor} (t + \underline{\underline{\delta}}) \xrightarrow{a} s \parallel (t + \underline{\underline{\delta}})$, and the only transition of the right-hand side is $\underline{\underline{a}} \cdot (s \parallel (t + \underline{\underline{\delta}})) \xrightarrow{a} s \parallel (t + \underline{\underline{\delta}})$, and note that $(s \parallel (t + \underline{\underline{\delta}}), s \parallel (t + \underline{\underline{\delta}})) \in R$. Finally, neither side satisfies the ID predicate: $\neg\mathrm{ID}(\underline{\underline{a}} \cdot s \mathbin{\|\!\lfloor} (t + \underline{\underline{\delta}}))$ and $\neg\mathrm{ID}(\underline{\underline{a}} \cdot (s \parallel (t + \underline{\underline{\delta}})))$.

Axiom DRTM5ID Take the relation:

$$R = \{(\sigma_{\mathrm{rel}}(s) \mathbin{\|\!\lfloor} (\nu_{\mathrm{rel}}(t) + \underline{\underline{\delta}}), \underline{\underline{\delta}}) \,|\, s,t \in C(\mathrm{PA}_{\mathrm{drt}})\}$$

We look at the transitions of both sides at the same time. Observe that there are no transitions possible on the left-hand side: $\sigma_{\mathrm{rel}}(s) \mathbin{\|\!\lfloor} (\nu_{\mathrm{rel}}(t) + \underline{\underline{\delta}}) \nrightarrow$. Also for the right-hand side there are no transitions possible: $\underline{\underline{\delta}} \nrightarrow$. Finally, neither side satisfies the ID predicate: $\neg\mathrm{ID}(\sigma_{\mathrm{rel}}(s) \mathbin{\|\!\lfloor} (\nu_{\mathrm{rel}}(t) + \underline{\underline{\delta}}))$ and $\neg\mathrm{ID}(\underline{\underline{\delta}})$ (note that $\neg\mathrm{ID}(\nu_{\mathrm{rel}}(t) + \underline{\underline{\delta}})$ even if $\mathrm{ID}(t)$).

Axiom DRTMID1 Take the relation:

$$R = \{(s \mathbin{\|\!\lfloor} \dot{\delta}, \dot{\delta}) \,|\, s \in C(\mathrm{PA}_{\mathrm{drt}})\}$$

We look at the transitions of both sides at the same time. Observe that there are no transitions possible on the left-hand side: $s \mathbin{\|\!\lfloor} \dot{\delta} \nrightarrow$. Also for the right-hand side there are no transitions possible: $\dot{\delta} \nrightarrow$. Finally, both sides satisfy the ID predicate: $\mathrm{ID}(s \mathbin{\|\!\lfloor} \dot{\delta})$ and $\mathrm{ID}(\dot{\delta})$ (note that $\mathrm{ID}(s \mathbin{\|\!\lfloor} \dot{\delta})$ even if $\neg\mathrm{ID}(s)$).

Axiom DRTMID2 Take the relation:

$$R = \{(\dot{\delta}, s \mathbin{\|\!\lfloor} \dot{\delta}) \,|\, s \in C(\mathrm{PA}_{\mathrm{drt}})\}$$

This case is treated symmetrically to the previous case.

■

Remark 3.6.12 (Soundness of $\mathrm{PA_{drt}}$)
Soundness of a somewhat different version of $\mathrm{PA_{drt}}$ is also claimed (without proof) in Section 3.9 of [10].

Theorem 3.6.13 (Conservativity of $\mathrm{PA^+_{drt}}$ with respect to $\mathrm{BPA^+_{drt}}$)
The equational specification PA^+_{drt} is a conservative extension of the equational specification BPA^+_{drt}.

Proof In order to prove conservativity it is sufficient to verify that the following conditions are satisfied:

(i). Bisimulation equivalence is definable in terms of predicate and relation symbols only,

(ii). $\mathrm{BPA^+_{drt}}$ is a complete axiomatization with respect to the bisimulation equivalence model induced by $T(\mathrm{BPA_{drt}})$ (see Theorem 2.8.22),

(iii). $\mathrm{PA^+_{drt}}$ is a sound axiomatization with respect to the bisimulation equivalence model induced by $T(\mathrm{PA_{drt}})$ (see Theorem 3.6.11),

(iv). $T(\mathrm{PA_{drt}})$ is an operationally conservative extension of $T(\mathrm{BPA_{drt}})$.

And in order for $T(\mathrm{PA_{drt}})$ to be an operationally conservative extension of $T(\mathrm{BPA_{drt}})$ we must verify the following conditions:

(i). $T(\mathrm{BPA_{drt}})$ is a pure, well-founded term deduction system in path format,

(ii). $T(\mathrm{PA_{drt}})$ is a term deduction system in path format,

(iii). $T(\mathrm{BPA_{drt}}) \oplus T(\mathrm{PA_{drt}})$ is defined.

That the above properties hold can be trivially checked from the relevant definitions. ■

Theorem 3.6.14 (Completeness of $\mathrm{PA^+_{drt}}$)
The equational specification PA^+_{drt} is a complete axiomatization of the set of closed PA_{drt} terms modulo bisimulation equivalence.

Proof By Verhoef's General Completeness Theorem (see [25], or Theorem 2.4.26 of [13]) this follows immediately from:

(i). $\mathrm{PA^+_{drt}}$ has the elimination property for $\mathrm{BPA_{drt}}$ (see Theorem 3.6.8),

(ii). $\mathrm{PA^+_{drt}}$ is a conservative extension of $\mathrm{BPA^+_{drt}}$ (see Theorem 3.6.13).

■

Remark 3.6.15 (Completeness of $\mathrm{PA_{drt}}$)
Completeness of a somewhat different version of $\mathrm{PA_{drt}}$ is also claimed (without proof) in Section 3.9 of [10].

Definition 3.6.16 (Axioms for the Ultimate Start Delay and Merge)
We define Axioms USD6 and USD7 for the ultimate start delay as given in Table 36 on the next page. Note that they precisely correspond to the equalities of Proposition 3.6.7(vi) and (vii).

$$a \mathbin{\|\!\!\lfloor} \lfloor x \rfloor^{\omega} = a \cdot \lfloor x \rfloor^{\omega} \qquad \text{USD6}$$

$$a \cdot x \mathbin{\|\!\!\lfloor} \lfloor y \rfloor^{\omega} = a \cdot (x \parallel \lfloor y \rfloor^{\omega}) \qquad \text{USD7}$$

Table 36: Additional axioms for the ultimate start delay and merge.

Corollary 3.6.17 (Soundness of $\mathrm{PA}_{\mathrm{drt}}$ + USD1–USD7)
The set of closed PA_{drt} terms modulo bisimulation equivalence is a model of PA_{drt} + USD1–USD7.

Proof This now follows directly from the soundness of $\mathrm{PA}^{+}_{\mathrm{drt}}$ (see Theorem 3.6.11 on page 106) and the fact that Axioms USD1–USD7 are derivable in $\mathrm{PA}^{+}_{\mathrm{drt}}$ (see Proposition 3.6.7 on page 101). ■

Corollary 3.6.18 (Completeness of $\mathrm{PA}_{\mathrm{drt}}$ + USD1–USD7)
If we add Axioms USD1–USD4 of Table 18 on page 45, Axiom USD5 of Table 23 on page 68, and Axioms USD6–USD7 of Table 36 to PA_{drt}, we again have a complete axiomatization of the set of closed PA_{drt} terms modulo bisimulation equivalence.

Proof Careful inspection of the dependencies between the proofs in this section reveals that the proof of Theorem 3.6.14 only relies upon RSP(USD) to ensure Proposition 3.6.7(i)–(vii). So, we obviously do not need RSP(USD) anymore if we add the corresponding Axioms USD1–USD7. Note that in this way we get a purely equational axiomatization (i.e. without conditional axioms or principles). ■

3.7 Soundness and Completeness of $\mathrm{ACP}^{+}_{\mathrm{drt}}$

Definition 3.7.1 (Signature of $\mathrm{ACP}_{\mathrm{drt}}$)
The signature of $\mathrm{ACP}_{\mathrm{drt}}$ consists of the *undelayable atomic actions* $\{\underline{\underline{a}} | a \in A\}$, the *delayable atomic actions* $\{\underline{a} | a \in A\}$, the *undelayable deadlock constant* $\underline{\underline{\delta}}$, the *delayable deadlock constant* $\underline{\delta}$, the *immediate deadlock constant* $\dot{\delta}$, the *alternative composition operator* $+$, the *sequential composition operator* $\cdot$, the *time unit delay operator* σ_{rel}, the *"now" operator* ν_{rel}, the *unbounded start delay operator* $\lfloor\ \rfloor^{\omega}$, the *(communicating) merge operator* $\parallel$, the *left merge operator* $\mathbin{\|\!\!\lfloor}$, and the *communication merge operator* $|$.

Definition 3.7.2 (Axioms of $\mathrm{ACP}_{\mathrm{drt}}$)
The process algebra $\mathrm{ACP}_{\mathrm{drt}}$ is axiomatized by the axioms of $\mathrm{PA}_{\mathrm{drt}}$ given in Definition 3.6.2 on page 100 *minus* Axiom DRTM1, *plus* Axioms DRTCM1–DRTCM5, DRTCM12–13, and DRTCF1–DRTCF2 shown in Table 29 on page 84, Axioms DRTCM6–DRTCM7 shown in Table 30 on page 85, and Axioms DRTMID3–DRTMID4 and DRTCM6ID–DRTCM7ID shown in Table 37 on the following page: $\mathrm{ACP}_{\mathrm{drt}}$ = A1–A5 + A6ID + A7ID + DRT1–DRT5 + DRTSID + DCS1–DCS4 + DCSID + ATS + USD + DRTM2ID–DRTM3ID + DRTM4 + DRTM5ID + DRTM6 + DRTCM1–DRTCM5 + DRTCM6ID–DRTCM7ID + DRTCM12–DRTCM13 + DRTCF1–DRTCF2 + DRTMID1–DRTMID4.

$$\sigma_{\mathrm{rel}}(x) \mid (\nu_{\mathrm{rel}}(y) + \underline{\underline{\delta}}) = \underline{\underline{\delta}} \qquad \text{DRTCM6ID}$$
$$(\nu_{\mathrm{rel}}(x) + \underline{\underline{\delta}}) \mid \sigma_{\mathrm{rel}}(y) = \underline{\underline{\delta}} \qquad \text{DRTCM7ID}$$
$$x \mid \dot{\delta} = \dot{\delta} \qquad \text{DRTMID3}$$
$$\dot{\delta} \mid x = \dot{\delta} \qquad \text{DRTMID4}$$

Table 37: Additional axioms for $\mathrm{ACP}_{\mathrm{drt}}$.

Definition 3.7.3 (Semantics of $\mathbf{ACP_{drt}}$)
The semantics of $\mathrm{ACP}_{\mathrm{drt}}$ are given by the term deduction system $T(\mathrm{ACP}_{\mathrm{drt}})$ induced by the deduction rules for $\mathrm{PA}_{\mathrm{drt}}$ given in Definition 3.6.3 on page 100, the deduction rules for the communication merge shown in Table 31 on page 85, and the additional deduction rules for $\mathrm{ACP}_{\mathrm{drt}}$ shown in Table 38.

$$\frac{\mathrm{ID}(x)}{\mathrm{ID}(x \mid y)} \qquad \frac{\mathrm{ID}(y)}{\mathrm{ID}(x \mid y)}$$

Table 38: Additional deduction rules for $\mathrm{ACP}_{\mathrm{drt}}$.

Definition 3.7.4 (Bisimulation and Bisimulation Model for $\mathbf{ACP_{drt}}$)
Bisimulation for $\mathrm{ACP}_{\mathrm{drt}}$ and the corresponding bisimulation model are defined in the same way as for $\mathrm{BPA}^{-}_{\mathrm{drt}}$ and BPA respectively. Replace "$\mathrm{BPA}^{-}_{\mathrm{drt}}$" by "$\mathrm{ACP}_{\mathrm{drt}}$" in Definition 2.7.7 on page 46 and "BPA" by "$\mathrm{ACP}_{\mathrm{drt}}$" in Definition 2.2.11 on page 8.

Definition 3.7.5 (Basic Terms of $\mathbf{ACP_{drt}}$)
If we speak of basic terms in the context of $\mathrm{ACP}_{\mathrm{drt}}$, we mean $(\sigma, \underline{\underline{\delta}}, \delta, \dot{\delta})$-basic terms as defined in Definition 2.8.7 on page 54.

Definition 3.7.6 (Number of Symbols of an $\mathbf{ACP_{drt}}$ term)
We define $n(x)$, the number of symbols of x, inductively as follows:

(i). We define $n(\dot{\delta}) = 1$,

(ii). for $a \in A_\delta$, we define $n(\underline{\underline{a}}) = n(a) = 1$,

(iii). for closed $\mathrm{ACP}_{\mathrm{drt}}$ terms x and y, we define $n(x + y) = n(x \cdot y) = n(x \parallel y) = n(x \mathbin{\|\!\_} y) = n(x \mid y) = n(x) + n(y) + 1$,

(iv). for a closed $\mathrm{ACP}_{\mathrm{drt}}$ term x, we define $n(\sigma_{\mathrm{rel}}(x)) = n(\nu_{\mathrm{rel}}(x)) = n(\lfloor x \rfloor^{\omega}) = n(x) + 1$.

Proposition 3.7.7 (Properties of $\mathbf{ACP^{+}_{drt}}$, Part I)
For ACP_{drt} terms x and y, and any $a \in A_\delta$, we have the following equalities:

(i). $ACP^{+}_{drt} \vdash \lfloor a \rfloor^{\omega} = a$

(ii). $ACP^{+}_{drt} \vdash \lfloor x \cdot y \rfloor^{\omega} = \lfloor x \rfloor^{\omega} \cdot y$

(iii). $ACP^{+}_{drt} \vdash \lfloor x + y \rfloor^{\omega} = \lfloor x \rfloor^{\omega} + \lfloor y \rfloor^{\omega}$

(iv). $ACP^{+}_{drt} \vdash \lfloor \sigma_{rel}(x) \rfloor^{\omega} = \delta$

(v). $ACP^{+}_{drt} \vdash \lfloor \dot{\delta} \rfloor^{\omega} = \delta$

(vi). $ACP^{+}_{drt} \vdash a \mathbin{\|\!\_} \lfloor x \rfloor^{\omega} = a \cdot \lfloor x \rfloor^{\omega}$

(vii). $ACP^{+}_{drt} \vdash a \cdot x \mathbin{\|\!\_} \lfloor y \rfloor^{\omega} = a \cdot (x \parallel \lfloor y \rfloor^{\omega})$

(viii). $ACP_{drt} \vdash \nu_{rel}(a) = \underline{\underline{a}}$

(ix). $ACP_{drt} \vdash \lfloor x \rfloor^{\omega} + \underline{\underline{\delta}} = \lfloor x \rfloor^{\omega}$

Proof The proofs for these equalities given in Proposition 3.6.7 on page 101, with respect to $\mathrm{PA}_{\mathrm{drt}}$, remain valid in the setting of $\mathrm{ACP}_{\mathrm{drt}}$, as can be easily checked. ∎

Proposition 3.7.8 (Properties of $\mathrm{ACP}^{+}_{\mathbf{drt}}$, Part II)
For ACP_{drt} terms x and y and any $a, b, c \in A_{\delta}$, we have the following equalities:

(i). $ACP^{+}_{drt} \vdash a \mid b = c \qquad \text{if } \gamma(a,b) = c \neq \delta$

(ii). $ACP^{+}_{drt} \vdash a \mid b = \delta \qquad \text{if } \gamma(a,b) = \delta$

(iii). $ACP^{+}_{drt} \vdash a \mid b \cdot x = (a \mid b) \cdot x$

(iv). $ACP^{+}_{drt} \vdash a \cdot x \mid b = (a \mid b) \cdot x$

(v). $ACP^{+}_{drt} \vdash a \cdot x \mid b \cdot y = (a \mid b) \cdot (x \parallel y)$

Proof

(i). Consider the following computation:

$$\begin{aligned}
\mathrm{ACP}_{\mathrm{drt}} \vdash a \mid b &= \lfloor \underline{\underline{a}} \rfloor^{\omega} \mid \lfloor \underline{\underline{b}} \rfloor^{\omega} \\
&= (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega})) \mid (\nu_{\mathrm{rel}}(\underline{\underline{b}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{b}} \rfloor^{\omega})) \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \mid (\underline{\underline{b}} + \sigma_{\mathrm{rel}}(b)) \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} + \underline{\underline{a}} \mid \sigma_{\mathrm{rel}}(b) + \sigma_{\mathrm{rel}}(a) \mid \underline{\underline{b}} + \sigma_{\mathrm{rel}}(a) \mid \sigma_{\mathrm{rel}}(b) \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} + \nu_{\mathrm{rel}}(\underline{\underline{a}}) \mid \sigma_{\mathrm{rel}}(b) + \sigma_{\mathrm{rel}}(a) \mid \nu_{\mathrm{rel}}(\underline{\underline{b}}) + \sigma_{\mathrm{rel}}(a) \mid \sigma_{\mathrm{rel}}(b) \\
&= \underline{\underline{c}} + \underline{\underline{\delta}} + \underline{\underline{\delta}} + \sigma_{\mathrm{rel}}(a \mid b) \\
&= \underline{\underline{c}} + \sigma_{\mathrm{rel}}(a \mid b) \\
&= \nu_{\mathrm{rel}}(\underline{\underline{c}}) + \sigma_{\mathrm{rel}}(a \mid b)
\end{aligned}$$

Using RSP(USD), we obtain:

$$\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash a \mid b = \lfloor \underline{\underline{c}} \rfloor^{\omega} = c$$

(ii). This case is treated like the previous one, but with c replaced by δ.

(iii). Consider the following computation:

$$
\begin{aligned}
\mathrm{ACP}_{\mathrm{drt}} \vdash a \mid b \cdot x &= \lfloor \underline{\underline{a}} \rfloor^{\omega} \mid \lfloor \underline{\underline{b}} \rfloor^{\omega} \cdot x \\
&= (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega})) \mid (\nu_{\mathrm{rel}}(\underline{\underline{b}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{b}} \rfloor^{\omega})) \cdot x \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \mid (\underline{\underline{b}} + \sigma_{\mathrm{rel}}(b)) \cdot x \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \mid (\underline{\underline{b}} \cdot x + \sigma_{\mathrm{rel}}(b) \cdot x) \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} \cdot x + \underline{\underline{a}} \mid \sigma_{\mathrm{rel}}(b) \cdot x + {} \\
&\qquad \sigma_{\mathrm{rel}}(a) \mid \underline{\underline{b}} \cdot x + \sigma_{\mathrm{rel}}(a) \mid \sigma_{\mathrm{rel}}(b) \cdot x \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} \cdot x + \nu_{\mathrm{rel}}(\underline{\underline{a}}) \mid \sigma_{\mathrm{rel}}(b) \cdot x + {} \\
&\qquad \sigma_{\mathrm{rel}}(a) \mid \nu_{\mathrm{rel}}(\underline{\underline{b}}) \cdot x + \sigma_{\mathrm{rel}}(a) \mid \sigma_{\mathrm{rel}}(b) \cdot x \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} \cdot x + \nu_{\mathrm{rel}}(\underline{\underline{a}}) \mid \sigma_{\mathrm{rel}}(b \cdot x) + {} \\
&\qquad \sigma_{\mathrm{rel}}(a) \mid \nu_{\mathrm{rel}}(\underline{\underline{b}} \cdot x) + \sigma_{\mathrm{rel}}(a) \mid \sigma_{\mathrm{rel}}(b \cdot x) \\
&= \underline{\underline{\gamma(a,b)}} \cdot x + \underline{\underline{\delta}} + \underline{\underline{\delta}} + \sigma_{\mathrm{rel}}(a \mid b \cdot x) \\
&= \underline{\underline{\gamma(a,b)}} \cdot x + \sigma_{\mathrm{rel}}(a \mid b \cdot x) \\
&= \nu_{\mathrm{rel}}(\underline{\underline{\underline{\gamma(a,b)}}}) \cdot x + \sigma_{\mathrm{rel}}(a \mid b \cdot x) \\
&= \nu_{\mathrm{rel}}(\underline{\underline{\underline{\gamma(a,b)}}} \cdot x) + \sigma_{\mathrm{rel}}(a \mid b \cdot x)
\end{aligned}
$$

Using RSP(USD), we obtain:

$$
\begin{aligned}
\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash a \mid b \cdot x &= \lfloor \underline{\underline{\gamma(a,b)}} \cdot x \rfloor^{\omega} = \lfloor \underline{\underline{\gamma(a,b)}} \rfloor^{\omega} \cdot x = \gamma(a,b) \cdot x \\
&= (a \mid b) \cdot x
\end{aligned}
$$

(iv). This case is treated symmetrically to the previous case.

(v). Consider the following computation:

$$
\begin{aligned}
\mathrm{ACP}_{\mathrm{drt}} \vdash a \cdot x \mid b \cdot y &= \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot x \mid \lfloor \underline{\underline{b}} \rfloor^{\omega} \cdot y \\
&= (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega})) \cdot x \mid (\nu_{\mathrm{rel}}(\underline{\underline{b}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{b}} \rfloor^{\omega})) \cdot y \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \cdot x \mid (\underline{\underline{b}} + \sigma_{\mathrm{rel}}(b)) \cdot y \\
&= (\underline{\underline{a}} \cdot x + \sigma_{\mathrm{rel}}(a) \cdot x) \mid (\underline{\underline{b}} \cdot y + \sigma_{\mathrm{rel}}(b) \cdot y) \\
&= \underline{\underline{a}} \cdot x \mid \underline{\underline{b}} \cdot y + \underline{\underline{a}} \cdot x \mid \sigma_{\mathrm{rel}}(b) \cdot y + {} \\
&\qquad \sigma_{\mathrm{rel}}(a) \cdot x \mid \underline{\underline{b}} \cdot y + \sigma_{\mathrm{rel}}(a) \cdot x \mid \sigma_{\mathrm{rel}}(b) \cdot y \\
&= \underline{\underline{a}} \cdot x \mid \underline{\underline{b}} \cdot y + \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot x \mid \sigma_{\mathrm{rel}}(b) \cdot y + {} \\
&\qquad \sigma_{\mathrm{rel}}(a) \cdot x \mid \nu_{\mathrm{rel}}(\underline{\underline{b}}) \cdot y + \sigma_{\mathrm{rel}}(a) \cdot x \mid \sigma_{\mathrm{rel}}(b) \cdot y \\
&= \underline{\underline{a}} \cdot x \mid \underline{\underline{b}} \cdot y + \nu_{\mathrm{rel}}(\underline{\underline{a}} \cdot x) \mid \sigma_{\mathrm{rel}}(b \cdot y) + {} \\
&\qquad \sigma_{\mathrm{rel}}(a \cdot x) \mid \nu_{\mathrm{rel}}(\underline{\underline{b}} \cdot y) + \sigma_{\mathrm{rel}}(a \cdot x) \mid \sigma_{\mathrm{rel}}(b \cdot y) \\
&= \underline{\underline{\gamma(a,b)}} \cdot (x \parallel y) + \underline{\underline{\delta}} + \underline{\underline{\delta}} + \sigma_{\mathrm{rel}}(a \cdot x \mid b \cdot y) \\
&= \underline{\underline{\gamma(a,b)}} \cdot (x \parallel y) + \sigma_{\mathrm{rel}}(a \cdot x \mid b \cdot y) \\
&= \nu_{\mathrm{rel}}(\underline{\underline{\underline{\gamma(a,b)}}}) \cdot (x \parallel y) + \sigma_{\mathrm{rel}}(a \cdot x \mid b \cdot y) \\
&= \nu_{\mathrm{rel}}(\underline{\underline{\underline{\gamma(a,b)}}} \cdot (x \parallel y)) + \sigma_{\mathrm{rel}}(a \cdot x \mid b \cdot y)
\end{aligned}
$$

Using RSP(USD), we obtain:

$$
\begin{aligned}
\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash a \cdot x \mid b \cdot y &= \lfloor \underline{\underline{\gamma(a,b)}} \cdot (x \parallel y) \rfloor^{\omega} \\
&= \lfloor \underline{\underline{\gamma(a,b)}} \rfloor^{\omega} \cdot (x \parallel y) \\
&= \gamma(a,b) \cdot (x \parallel y) \\
&= (a \mid b) \cdot (x \parallel y)
\end{aligned}
$$

■

Remark 3.7.9 (Properties of $\mathbf{ACP}^{+}_{\mathbf{drt}}$, Part II)
Note that the equalities of Proposition 3.7.8 on page 111 are in a sense the "delayable reformulations" of Axioms DRTCF1–DRTCF2 and DRTCM2–DRTCM4 for the communication merge $\mid$. Such reformulations are however not possible for the axioms for the left merge $\mathbin{\|\!\lfloor}$. Take for example DRTM2ID; although we do have:

$$\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash \underline{\underline{a}} \mathbin{\|\!\lfloor} (x + \underline{\underline{\delta}}) = \underline{\underline{a}} \cdot (x + \underline{\underline{\delta}})$$

the "delayable" reformulation does not hold:

$$\mathrm{ACP}^{+}_{\mathrm{drt}} \nvdash a \mathbin{\|\!\lfloor} (x + \underline{\underline{\delta}}) = a \cdot (x + \underline{\underline{\delta}})$$

as can be seen by instantiating x with any x such that $x \overset{\sigma}{\nrightarrow}$. In that case, namely, $a \cdot (x + \underline{\underline{\delta}})$ can delay, while $a \mathbin{\|\!\lfloor} (x + \underline{\underline{\delta}})$ cannot, because x cannot.

Proposition 3.7.10 (Properties of $\mathbf{ACP}_{\mathbf{drt}}$, Part III)
For ACP_{drt} terms x and y, and any $a, b \in A_{\delta}$, we have the following equalities:

(i). $ACP_{drt} \vdash \underline{\underline{a}} \mid b = \underline{\underline{a}} \mid \underline{\underline{b}}$

(ii). $ACP_{drt} \vdash a \mid \underline{\underline{b}} = \underline{\underline{a}} \mid \underline{\underline{b}}$

(iii). $ACP_{drt} \vdash \underline{\underline{a}} \mid b \cdot x = (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot x$

(iv). $ACP_{drt} \vdash a \cdot x \mid \underline{\underline{b}} = (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot x$

(v). $ACP_{drt} \vdash a \mid \underline{\underline{b}} \cdot x = (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot x$

(vi). $ACP_{drt} \vdash \underline{\underline{a}} \cdot x \mid b = (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot x$

(vii). $ACP_{drt} \vdash \underline{\underline{a}} \cdot x \mid b \cdot y = (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot (x \parallel y)$

(viii). $ACP_{drt} \vdash a \cdot x \mid \underline{\underline{b}} \cdot y = (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot (x \parallel y)$

Proof

(i). Consider the following computation:

$$
\begin{aligned}
\mathrm{ACP}_{\mathrm{drt}} \vdash \underline{\underline{a}} \mid b &= \underline{\underline{a}} \mid \lfloor \underline{\underline{b}} \rfloor^{\omega} \\
&= \underline{\underline{a}} \mid (\nu_{\mathrm{rel}}(\underline{\underline{b}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{b}} \rfloor^{\omega}) \\
&= \underline{\underline{a}} \mid (\underline{\underline{b}} + \sigma_{\mathrm{rel}}(b)) \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} + \underline{\underline{a}} \mid \sigma_{\mathrm{rel}}(b) \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} + \nu_{\mathrm{rel}}(\underline{\underline{a}}) \mid \sigma_{\mathrm{rel}}(b) \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} + \underline{\underline{\delta}} \\
&= \underline{\underline{a}} \mid \underline{\underline{b}}
\end{aligned}
$$

(ii). This case is treated symmetrically to the previous case.

(iii). Consider the following computation:

$$
\begin{aligned}
\mathrm{ACP}_{\mathrm{drt}} \vdash \underline{\underline{a}} \mid b \cdot x &= \underline{\underline{a}} \mid \lfloor \underline{\underline{b}} \rfloor^{\omega} \cdot x \\
&= \underline{\underline{a}} \mid (\nu_{\mathrm{rel}}(\underline{\underline{b}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{b}} \rfloor^{\omega}) \cdot x \\
&= \underline{\underline{a}} \mid (\underline{\underline{b}} + \sigma_{\mathrm{rel}}(b)) \cdot x \\
&= \underline{\underline{a}} \mid (\underline{\underline{b}} \cdot x + \sigma_{\mathrm{rel}}(b) \cdot x) \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} \cdot x + \underline{\underline{a}} \mid \sigma_{\mathrm{rel}}(b) \cdot x \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} \cdot x + \nu_{\mathrm{rel}}(\underline{\underline{a}}) \mid \sigma_{\mathrm{rel}}(b \cdot x) \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} \cdot x + \underline{\underline{\delta}} \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} \cdot x \\
&= (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot x
\end{aligned}
$$

(iv). This case is treated symmetrically to the previous case.

(v). Consider the following computation:

$$
\begin{aligned}
\mathrm{ACP}_{\mathrm{drt}} \vdash a \mid \underline{\underline{b}} \cdot x &= \lfloor \underline{\underline{a}} \rfloor^{\omega} \mid \underline{\underline{b}} \cdot x \\
&= (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega}) \mid \underline{\underline{b}} \cdot x \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \mid \underline{\underline{b}} \cdot x \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} \cdot x + \sigma_{\mathrm{rel}}(a) \mid \underline{\underline{b}} \cdot x \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} \cdot x + \sigma_{\mathrm{rel}}(a) \mid \nu_{\mathrm{rel}}(\underline{\underline{b}}) \cdot x \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} \cdot x + \sigma_{\mathrm{rel}}(a) \mid \nu_{\mathrm{rel}}(\underline{\underline{b}} \cdot x) \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} \cdot x + \underline{\underline{\delta}} \\
&= \underline{\underline{a}} \mid \underline{\underline{b}} \cdot x \\
&= (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot x
\end{aligned}
$$

(vi). This case is treated symmetrically to the previous case.

(vii). Consider the following computation:

$$
\begin{aligned}
\mathrm{ACP}_{\mathrm{drt}} \vdash \underline{\underline{a}} \cdot x \mid b \cdot y &= \underline{\underline{a}} \cdot x \mid \lfloor \underline{\underline{b}} \rfloor^{\omega} \cdot y \\
&= \underline{\underline{a}} \cdot x \mid (\nu_{\mathrm{rel}}(\underline{\underline{b}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{b}} \rfloor^{\omega})) \cdot y \\
&= \underline{\underline{a}} \cdot x \mid (\underline{\underline{b}} + \sigma_{\mathrm{rel}}(b)) \cdot y \\
&= \underline{\underline{a}} \cdot x \mid (\underline{\underline{b}} \cdot y + \sigma_{\mathrm{rel}}(b) \cdot y) \\
&= \underline{\underline{a}} \cdot x \mid \underline{\underline{b}} \cdot y + \underline{\underline{a}} \cdot x \mid \sigma_{\mathrm{rel}}(b) \cdot y \\
&= \underline{\underline{a}} \cdot x \mid \underline{\underline{b}} \cdot y + \nu_{\mathrm{rel}}(\underline{\underline{a}}) \cdot x \mid \sigma_{\mathrm{rel}}(b) \cdot y \\
&= \underline{\underline{a}} \cdot x \mid \underline{\underline{b}} \cdot y + \nu_{\mathrm{rel}}(\underline{\underline{a}} \cdot x) \mid \sigma_{\mathrm{rel}}(b \cdot y) \\
&= \underline{\underline{a}} \cdot x \mid \underline{\underline{b}} \cdot y + \underline{\underline{\delta}} \\
&= \underline{\underline{a}} \cdot x \mid \underline{\underline{b}} \cdot y \\
&= (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot (x \parallel y)
\end{aligned}
$$

(viii). This case is treated symmetrically to the previous case.

$\blacksquare$

Theorem 3.7.11 (Elimination for ACP$^+_{\mathrm{drt}}$)
Let t be a closed ACP$_{drt}$ term. Then there is closed BPA$_{drt}$ term s such that ACP$^+_{drt} \vdash t = s$.

Proof Let t be a closed ACP$_{\mathrm{drt}}$ term. The theorem is proven by induction on $n(t)$ and case distinction on the general structure of t.

(i). $t \equiv \dot{\delta}$. Then t is a closed BPA$_{\mathrm{drt}}$ term.

(ii). $t \equiv \underline{\underline{a}}$ for some $a \in A_\delta$. Then t is a closed BPA$_{\mathrm{drt}}$ term.

(iii). $t \equiv a$ for some $a \in A_\delta$. Then t is a closed BPA$_{\mathrm{drt}}$ term.

(iv). $t \equiv t_1 + t_2$ for closed ACP$_{\mathrm{drt}}$ terms t_1 and t_2. By induction there are closed BPA$_{\mathrm{drt}}$ terms s_1 and s_2 such that ACP$^+_{\mathrm{drt}} \vdash t_1 = s_1$ and ACP$^+_{\mathrm{drt}} \vdash t_2 = s_2$. But then also ACP$^+_{\mathrm{drt}} \vdash t_1 + t_2 = s_1 + s_2$ and $s_1 + s_2$ is a closed BPA$_{\mathrm{drt}}$ term.

(v). $t \equiv t_1 \cdot t_2$ for closed ACP$_{\mathrm{drt}}$ terms t_1 and t_2. This case is treated analogously to case (ii).

(vi). $t \equiv \sigma_{\mathrm{rel}}(t_1)$ for a closed ACP$_{\mathrm{drt}}$ term t_1. This case is treated analogously to case (ii).

(vii). $t \equiv \nu_{\mathrm{rel}}(t_1)$ for a closed ACP$_{\mathrm{drt}}$ term t_1. This case is treated analogously to case (ii).

(viii). $t \equiv \lfloor t_1 \rfloor^\omega$ for a closed ACP$_{\mathrm{drt}}$ term t_1. This case is treated analogously to case (ii).

(ix). $t \equiv t_1 \mathbin{|\!\!\lfloor} t_2$ for closed ACP$_{\mathrm{drt}}$ terms t_1 and t_2. By induction there are closed BPA$_{\mathrm{drt}}$ terms s_1 and s_2 such that ACP$^+_{\mathrm{drt}} \vdash t_1 = s_1$ and ACP$^+_{\mathrm{drt}} \vdash t_2 = s_2$. By Theorem 2.8.16, the elimination theorem for BPA$_{\mathrm{drt}}$, there are basic terms r_1 and r_2 such that BPA$^+_{\mathrm{drt}} \vdash s_1 = r_1$ and BPA$^+_{\mathrm{drt}} \vdash s_2 = r_2$. But then also, ACP$^+_{\mathrm{drt}} \vdash t_1 = r_1$, ACP$^+_{\mathrm{drt}} \vdash t_2 = r_2$, and ACP$^+_{\mathrm{drt}} \vdash t_1 \mathbin{|\!\!\lfloor} t_2 = r_1 \mathbin{|\!\!\lfloor} r_2$. We proceed by induction on the structure of basic terms, and distinguish all possible cases for basic term r_1:

(a) $r_1 \equiv \dot{\delta}$. Then ACP$^+_{\mathrm{drt}} \vdash t_1 \mathbin{|\!\!\lfloor} t_2 = r_1 \mathbin{|\!\!\lfloor} r_2 = \dot{\delta} \mathbin{|\!\!\lfloor} r_2 = \dot{\delta}$, and $\dot{\delta}$ is a closed BPA$_{\mathrm{drt}}$ term.

(b) $r_1 \equiv \underline{\underline{a}}$ for some $a \in A_\delta$. Using Lemma 2.8.14 we distinguish two cases:

i. $r_2 = \dot{\delta}$. Then we have: ACP$^+_{\mathrm{drt}} \vdash t_1 \mathbin{|\!\!\lfloor} t_2 = r_1 \mathbin{|\!\!\lfloor} r_2 = r_1 \mathbin{|\!\!\lfloor} \dot{\delta} = \dot{\delta}$, and $\dot{\delta}$ is a closed BPA$_{\mathrm{drt}}$ term.

ii. $r_2 = r_2 + \underline{\underline{\delta}}$. Then we have: ACP$^+_{\mathrm{drt}} \vdash t_1 \mathbin{|\!\!\lfloor} t_2 = r_1 \mathbin{|\!\!\lfloor} r_2 = \underline{\underline{a}} \mathbin{|\!\!\lfloor} (r_2 + \underline{\underline{\delta}}) = \underline{\underline{a}} \cdot (r_2 + \underline{\underline{\delta}}) = \underline{\underline{a}} \cdot r_2$, and $\underline{\underline{a}} \cdot r_2$ is a closed BPA$_{\mathrm{drt}}$ term.

(c) $r_1 \equiv a$ for some $a \in A_\delta$. Using Lemma 2.8.13 we distinguish four cases:

i. $r_2 = \dot{\delta}$. Then we have: ACP$^+_{\mathrm{drt}} \vdash t_1 \mathbin{|\!\!\lfloor} t_2 = r_1 \mathbin{|\!\!\lfloor} r_2 = r_1 \mathbin{|\!\!\lfloor} \dot{\delta} = \dot{\delta}$, and $\dot{\delta}$ is a closed BPA$_{\mathrm{drt}}$ term.

ii. $r_2 = \nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}$. Then we have:

$$\begin{aligned}
\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mathbin{|\!\!\lfloor} t_2 &= r_1 \mathbin{|\!\!\lfloor} r_2 \\
&= a \mathbin{|\!\!\lfloor} r_2 \\
&= \lfloor \underline{\underline{a}} \rfloor^\omega \mathbin{|\!\!\lfloor} r_2
\end{aligned}$$

$$
\begin{aligned}
&= (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega})) \mathbin{\|\!\!\!\_} r_2 \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \mathbin{\|\!\!\!\_} r_2 \\
&= \underline{\underline{a}} \mathbin{\|\!\!\!\_} r_2 + \sigma_{\mathrm{rel}}(a) \mathbin{\|\!\!\!\_} r_2 \\
&= \underline{\underline{a}} \mathbin{\|\!\!\!\_} (\nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}) + \sigma_{\mathrm{rel}}(a) \mathbin{\|\!\!\!\_} (\nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}) \\
&= \underline{\underline{a}} \cdot (\nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}) + \underline{\underline{\delta}} \\
&= \underline{\underline{a}} \cdot r_2 + \underline{\underline{\delta}} \\
&= \underline{\underline{a}} \cdot r_2 + \underline{\underline{\delta}} \cdot r_2 \\
&= (\underline{\underline{a}} + \underline{\underline{\delta}}) \cdot r_2 \\
&= \underline{\underline{a}} \cdot r_2,
\end{aligned}
$$

and $\underline{\underline{a}} \cdot r_2$ is a closed $\mathrm{BPA_{drt}}$ term.

iii. $r_2 = \lfloor r_2 \rfloor^{\omega}$. Then, using Proposition 3.7.7(vi), we have: $\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\!\_} t_2 = r_1 \mathbin{\|\!\!\!\_} r_2 = a \mathbin{\|\!\!\!\_} \lfloor r_2 \rfloor^{\omega} = a \cdot \lfloor r_2 \rfloor^{\omega} = a \cdot r_2$, and $a \cdot r_2$ is a closed $\mathrm{BPA_{drt}}$ term.

iv. $r_2 = \nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')$ for a basic term r_2' such that $n(r_2') < n(r_2)$. Then we have:

$$
\begin{aligned}
\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\!\_} t_2 &= r_1 \mathbin{\|\!\!\!\_} r_2 \\
&= a \mathbin{\|\!\!\!\_} r_2 \\
&= \lfloor \underline{\underline{a}} \rfloor^{\omega} \mathbin{\|\!\!\!\_} r_2 \\
&= (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega})) \mathbin{\|\!\!\!\_} r_2 \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \mathbin{\|\!\!\!\_} r_2 \\
&= \underline{\underline{a}} \mathbin{\|\!\!\!\_} r_2 + \sigma_{\mathrm{rel}}(a) \mathbin{\|\!\!\!\_} r_2 \\
&= \underline{\underline{a}} \mathbin{\|\!\!\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) + {} \\
&\qquad \sigma_{\mathrm{rel}}(a) \mathbin{\|\!\!\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) \\
&= \underline{\underline{a}} \mathbin{\|\!\!\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2') + \underline{\underline{\delta}}) + \sigma_{\mathrm{rel}}(a \mathbin{\|\!\!\!\_} r_2') \\
&= \underline{\underline{a}} \cdot (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2') + \underline{\underline{\delta}}) + \sigma_{\mathrm{rel}}(a \mathbin{\|\!\!\!\_} r_2') \\
&= \underline{\underline{a}} \cdot (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) + \sigma_{\mathrm{rel}}(a \mathbin{\|\!\!\!\_} r_2') \\
&= \underline{\underline{a}} \cdot r_2 + \sigma_{\mathrm{rel}}(a \mathbin{\|\!\!\!\_} r_2').
\end{aligned}
$$

By the induction hypothesis there exists a closed $\mathrm{BPA_{drt}}$ term p such that $\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash a \mathbin{\|\!\!\!\_} r_2' = p$. Then, $\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\!\_} t_2 = \underline{\underline{a}} \cdot r_2 + \sigma_{\mathrm{rel}}(a \mathbin{\|\!\!\!\_} r_2') = \underline{\underline{a}} \cdot r_2 + \sigma_{\mathrm{rel}}(p)$, and $\underline{\underline{a}} \cdot r_2 + \sigma_{\mathrm{rel}}(p)$ is a closed $\mathrm{BPA_{drt}}$ term.

(d) $r_1 \equiv \underline{\underline{a}} \cdot r_1'$ for some $a \in A_\delta$ and basic term r_1'. Using Lemma 2.8.14 we distinguish two cases:

i. $r_2 = \dot{\delta}$. Then we have: $\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\!\_} t_2 = r_1 \mathbin{\|\!\!\!\_} r_2 = r_1 \mathbin{\|\!\!\!\_} \dot{\delta} = \dot{\delta}$, and $\dot{\delta}$ is a closed $\mathrm{BPA_{drt}}$ term.

ii. $r_2 = r_2 + \underline{\underline{\delta}}$. Then we have: $\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\!\_} t_2 = r_1 \mathbin{\|\!\!\!\_} r_2 = \underline{\underline{a}} \cdot r_1' \mathbin{\|\!\!\!\_} (r_2 + \underline{\underline{\delta}}) = \underline{\underline{a}} \cdot (r_1' \parallel (r_2 + \underline{\underline{\delta}})) = \underline{\underline{a}} \cdot (r_1' \parallel r_2)$. By the induction hypothesis there exists a closed $\mathrm{BPA_{drt}}$ term p such that $\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash r_1' \parallel r_2 = p$. Then, $\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\!\!\_} t_2 = \underline{\underline{a}} \cdot (r_1' \parallel r_2) = \underline{\underline{a}} \cdot p$, and $\underline{\underline{a}} \cdot p$ is a closed $\mathrm{BPA_{drt}}$ term.

(e) $r_1 \equiv a \cdot r_1'$ for some $a \in A_\delta$ and basic term r_1'. Using Lemma 2.8.13 we distinguish four cases:

i. $r_2 = \dot{\delta}$. Then we have: $\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1 \mathbin{\|\!\_} r_2 = r_1 \mathbin{\|\!\_} \dot{\delta} = \dot{\delta}$, and $\dot{\delta}$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

ii. $r_2 = \nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}$. Then we have:

$$
\begin{aligned}
\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 &= r_1 \mathbin{\|\!\_} r_2 \\
&= a \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega})) \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= (\underline{\underline{a}} \cdot r_1' + \sigma_{\mathrm{rel}}(a) \cdot r_1') \mathbin{\|\!\_} r_2 \\
&= (\underline{\underline{a}} \cdot r_1' + \sigma_{\mathrm{rel}}(a \cdot r_1')) \mathbin{\|\!\_} r_2 \\
&= \underline{\underline{a}} \cdot r_1' \mathbin{\|\!\_} r_2 + \sigma_{\mathrm{rel}}(a \cdot r_1') \mathbin{\|\!\_} r_2 \\
&= \underline{\underline{a}} \cdot r_1' \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}) + \sigma_{\mathrm{rel}}(a \cdot r_1') \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}) \\
&= \underline{\underline{a}} \cdot (r_1' \parallel (\nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}})) + \underline{\underline{\delta}} \\
&= \underline{\underline{a}} \cdot (r_1' \parallel r_2) + \underline{\underline{\delta}} \\
&= \underline{\underline{a}} \cdot (r_1' \parallel r_2) + \underline{\underline{\delta}} \cdot (r_1' \parallel r_2) \\
&= (\underline{\underline{a}} + \underline{\underline{\delta}}) \cdot (r_1' \parallel r_2) \\
&= \underline{\underline{a}} \cdot (r_1' \parallel r_2).
\end{aligned}
$$

By the induction hypothesis there exists a closed $\mathrm{BPA}_{\mathrm{drt}}$ term p such that $\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash r_1' \parallel r_2 = p$. Then, $\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = \underline{\underline{a}} \cdot (r_1' \parallel r_2) = \underline{\underline{a}} \cdot p$, and $\underline{\underline{a}} \cdot p$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

iii. $r_2 = \lfloor r_2 \rfloor^{\omega}$. Then, using Proposition 3.7.7(vii), we have: $\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1 \mathbin{\|\!\_} r_2 = a \cdot r_1' \mathbin{\|\!\_} \lfloor r_2 \rfloor^{\omega} = a \cdot (r_1' \parallel \lfloor r_2 \rfloor^{\omega}) = a \cdot (r_1' \parallel r_2)$. By the induction hypothesis there exists a closed $\mathrm{BPA}_{\mathrm{drt}}$ term p such that $\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash r_1' \parallel r_2 = p$. Then, $\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = a \cdot (r_1' \parallel r_2) = a \cdot p$, and $a \cdot p$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

iv. $r_2 = \nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')$ for a basic term r_2' such that $n(r_2') < n(r_2)$. Then we have:

$$
\begin{aligned}
\mathrm{ACP}^{+}_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 &= r_1 \mathbin{\|\!\_} r_2 \\
&= a \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= (\nu_{\mathrm{rel}}(\underline{\underline{a}}) + \sigma_{\mathrm{rel}}(\lfloor \underline{\underline{a}} \rfloor^{\omega})) \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= (\underline{\underline{a}} + \sigma_{\mathrm{rel}}(a)) \cdot r_1' \mathbin{\|\!\_} r_2 \\
&= (\underline{\underline{a}} \cdot r_1' + \sigma_{\mathrm{rel}}(a) \cdot r_1') \mathbin{\|\!\_} r_2 \\
&= (\underline{\underline{a}} \cdot r_1' + \sigma_{\mathrm{rel}}(a \cdot r_1')) \mathbin{\|\!\_} r_2 \\
&= \underline{\underline{a}} \cdot r_1' \mathbin{\|\!\_} r_2 + \sigma_{\mathrm{rel}}(a \cdot r_1') \mathbin{\|\!\_} r_2 \\
&= \underline{\underline{a}} \cdot r_1' \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) + \\
&\qquad \sigma_{\mathrm{rel}}(a \cdot r_1') \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) \\
&= \underline{\underline{a}} \cdot r_1' \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2') + \underline{\underline{\delta}}) + \\
&\qquad \sigma_{\mathrm{rel}}(a \cdot r_1') \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2'))
\end{aligned}
$$

$$= \underline{\underline{a}} \cdot (r_1' \parallel (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2') + \underline{\underline{\delta}})) + \sigma_{\mathrm{rel}}(a \cdot r_1' \mathbin{\|\!\_} r_2')$$
$$= \underline{\underline{a}} \cdot (r_1' \parallel (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2'))) + \sigma_{\mathrm{rel}}(r_1 \mathbin{\|\!\_} r_2')$$
$$= \underline{\underline{a}} \cdot (r_1' \parallel r_2) + \sigma_{\mathrm{rel}}(r_1 \mathbin{\|\!\_} r_2').$$

By the induction hypothesis there exist closed $\mathrm{BPA}_{\mathrm{drt}}$ terms p_1 and p_2 such that $\mathrm{ACP}^+_{\mathrm{drt}} \vdash r_1' \parallel r_2 = p_1$ and $\mathrm{ACP}^+_{\mathrm{drt}} \vdash r_1 \mathbin{\|\!\_} r_2' = p_2$. Then, $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = \underline{\underline{a}} \cdot (r_1' \parallel r_2) + \sigma_{\mathrm{rel}}(r_1 \mathbin{\|\!\_} r_2') = \underline{\underline{a}} \cdot p_1 + \sigma_{\mathrm{rel}}(p_2)$, and $\underline{\underline{a}} \cdot p_1 + \sigma_{\mathrm{rel}}(p_2)$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(f) $r_1 \equiv r_1' + r_1''$ for basic terms r_1' and r_1''. Then $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1 \mathbin{\|\!\_} r_2 = (r_1' + r_1'') \mathbin{\|\!\_} r_2 = r_1' \mathbin{\|\!\_} r_2 + r_1'' \mathbin{\|\!\_} r_2$. By induction there exist closed $\mathrm{BPA}_{\mathrm{drt}}$ terms p_1 and p_2 such that $\mathrm{ACP}^+_{\mathrm{drt}} \vdash r_1' \mathbin{\|\!\_} r_2 = p_1$ and $\mathrm{ACP}^+_{\mathrm{drt}} \vdash r_1'' \mathbin{\|\!\_} r_2 = p_2$. Then also $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1' \mathbin{\|\!\_} r_2 + r_1'' \mathbin{\|\!\_} r_2 = p_1 + p_2$, and $p_1 + p_2$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(g) $r_1 \equiv \sigma_{\mathrm{rel}}(r_1')$ for a basic term r_1'. Using Lemma 2.8.13 we distinguish four cases:

i. $r_2 = \dot{\delta}$. Then we have: $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1 \mathbin{\|\!\_} r_2 = a \mathbin{\|\!\_} \dot{\delta} = \dot{\delta}$, and $\dot{\delta}$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

ii. $r_2 = \nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}$. Then $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1 \mathbin{\|\!\_} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}) = \underline{\underline{\delta}}$, and $\underline{\underline{\delta}}$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

iii. $r_2 = \lfloor r_2 \rfloor^{\omega}$. Then we have: $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1 \mathbin{\|\!\_} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{\|\!\_} \lfloor r_2 \rfloor^{\omega} = \sigma_{\mathrm{rel}}(r_1') \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(\lfloor r_2 \rfloor^{\omega}) = \sigma_{\mathrm{rel}}(r_1' \mathbin{\|\!\_} \lfloor r_2 \rfloor^{\omega}) = \sigma_{\mathrm{rel}}(r_1' \mathbin{\|\!\_} r_2)$. By the induction hypothesis there exists a closed $\mathrm{BPA}_{\mathrm{drt}}$ term p such that $\mathrm{ACP}^+_{\mathrm{drt}} \vdash r_1' \parallel r_2 = p$. Then, $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = \sigma_{\mathrm{rel}}(r_1' \parallel r_2) = \sigma_{\mathrm{rel}}(p)$, and $\sigma_{\mathrm{rel}}(p)$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

iv. $r_2 = \nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')$ for a basic term r_2' such that $n(r_2') < n(r_2)$. Then we have: $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = r_1 \mathbin{\|\!\_} r_2 = \sigma_{\mathrm{rel}}(r_1') \mathbin{\|\!\_} (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) = \sigma_{\mathrm{rel}}(r_1' \mathbin{\|\!\_} r_2')$. By the induction hypothesis there is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term p such that $\mathrm{ACP}^+_{\mathrm{drt}} \vdash r_1' \mathbin{\|\!\_} r_2' = p$. Then, $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mathbin{\|\!\_} t_2 = \sigma_{\mathrm{rel}}(r_1' \mathbin{\|\!\_} r_2') = \sigma_{\mathrm{rel}}(p)$, and $\sigma_{\mathrm{rel}}(p)$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(x). $t \equiv t_1 \mid t_2$ for closed $\mathrm{ACP}_{\mathrm{drt}}$ terms t_1 and t_2. By induction there are closed $\mathrm{BPA}_{\mathrm{drt}}$ terms s_1 and s_2 such that $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 = s_1$ and $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_2 = s_2$. By Theorem 2.8.16, the elimination theorem for $\mathrm{BPA}_{\mathrm{drt}}$, there are basic terms r_1 and r_2 such that $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s_1 = r_1$ and $\mathrm{BPA}^+_{\mathrm{drt}} \vdash s_2 = r_2$. But then also, $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 = r_1$, $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_2 = r_2$, and $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2$. We prove this case by simultaneous induction on the structure of basic terms r_1 and r_2. We examine all possible cases (of which there are in total 49, some of which can be treated simultaneously, reducing our task to "just" 22 cases):

(a) $r_1 \equiv \dot{\delta}$ and r_2 is of arbitrary form. Then $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \dot{\delta} \mid r_2 = \dot{\delta}$, and $\dot{\delta}$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(b) r_1 is of arbitrary form and $r_2 \equiv \dot{\delta}$. This case is treated symmetrically to the previous case.

(c) $r_1 \equiv \underline{\underline{a}}$ and $r_2 \equiv \underline{\underline{b}}$ for some $a, b \in A_{\delta}$. Suppose that $\gamma(a,b) = c$. Then we have $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \underline{\underline{a}} \mid \underline{\underline{b}} = \underline{\underline{c}}$, and $\underline{\underline{c}}$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(d) $r_1 \equiv \underline{\underline{a}}$ and $r_2 \equiv b$ for some $a, b \in A_\delta$. Suppose that $\gamma(a,b) = c$. Then we have $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \underline{\underline{a}} \mid b = \underline{\underline{c}}$, and $\underline{\underline{c}}$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(e) $r_1 \equiv a$ and $r_2 \equiv \underline{\underline{b}}$ for some $a, b \in A_\delta$. This case is treated symmetrically to the previous case.

(f) $r_1 \equiv a$ and $r_2 \equiv b$ for some $a, b \in A_\delta$. Suppose that $\gamma(a,b) = c$. Then we have $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = a \mid b = c$, and c is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(g) $r_1 \equiv \underline{\underline{a}}$ and $r_2 \equiv \underline{\underline{b}} \cdot r_2'$ for some $a, b \in A_\delta$ and some basic term r_2'. Suppose that $\gamma(a,b) = c$. Then we have $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \underline{\underline{a}} \mid \underline{\underline{b}} \cdot r_2' = \underline{\underline{c}} \cdot r_2'$, and $\underline{\underline{c}} \cdot r_2'$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(h) $r_1 \equiv \underline{\underline{a}} \cdot r_1'$ and $r_2 \equiv \underline{\underline{b}}$ for some $a, b \in A_\delta$ and some basic term r_1'. This case is treated symmetrically to the previous case.

(i) $r_1 \equiv \underline{\underline{a}}$ and $r_2 \equiv b \cdot r_2'$ for some $a, b \in A_\delta$ and some basic term r_2'. Suppose that $\gamma(a,b) = c$. Then we have $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \underline{\underline{a}} \mid b \cdot r_2' = \underline{\underline{c}} \cdot r_2'$, and $\underline{\underline{c}} \cdot r_2'$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(j) $r_1 \equiv a \cdot r_1'$ and $r_2 \equiv \underline{\underline{b}}$ for some $a, b \in A_\delta$ and some basic term r_1'. This case is treated symmetrically to the previous case.

(k) $r_1 \equiv a$ and $r_2 \equiv \underline{\underline{b}} \cdot r_2'$ for some $a, b \in A_\delta$ and some basic term r_2'. Suppose that $\gamma(a,b) = c$. Then we have $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = a \mid \underline{\underline{b}} \cdot r_2' = \underline{\underline{c}} \cdot r_2'$, and $\underline{\underline{c}} \cdot r_2'$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(l) $r_1 \equiv \underline{\underline{a}} \cdot r_1'$ and $r_2 \equiv b$ for some $a, b \in A_\delta$ and some basic term r_1'. This case is treated symmetrically to the previous case.

(m) $r_1 \equiv a$ and $r_2 \equiv b \cdot r_2'$ for some $a, b \in A_\delta$ and some basic term r_2'. Suppose that $\gamma(a,b) = c$. Then we have $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = a \mid b \cdot r_2' = c \cdot r_2'$, and $c \cdot r_2'$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(n) $r_1 \equiv a \cdot r_1'$ and $r_2 \equiv b$ for some $a, b \in A_\delta$ and some basic term r_1'. This case is treated symmetrically to the previous case.

(o) $r_1 \equiv \underline{\underline{a}} \cdot r_1'$ and $r_2 \equiv \underline{\underline{b}} \cdot r_2'$ for some $a, b \in A_\delta$ and some basic terms r_1' and r_2'. Suppose that $\gamma(a,b) = c$. Then we have $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \underline{\underline{a}} \cdot r_1' \mid \underline{\underline{b}} \cdot r_2' = \underline{\underline{c}} \cdot (r_1' \parallel r_2')$. By the induction hypothesis there exists a closed $\mathrm{BPA}_{\mathrm{drt}}$ term s' such that $\mathrm{ACP}^+_{\mathrm{drt}} \vdash r_1' \parallel r_2' = s'$. So $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = \underline{\underline{c}} \cdot (r_1' \parallel r_2') = \underline{\underline{c}} \cdot s'$, and $\underline{\underline{c}} \cdot s'$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(p) $r_1 \equiv \underline{\underline{a}} \cdot r_1'$ and $r_2 \equiv b \cdot r_2'$ for some $a, b \in A_\delta$ and some basic terms r_1' and r_2'. Suppose that $\gamma(a,b) = c$. Then we have $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \underline{\underline{a}} \cdot r_1' \mid b \cdot r_2' = \underline{\underline{c}} \cdot (r_1' \parallel r_2')$. By the induction hypothesis there exists a closed $\mathrm{BPA}_{\mathrm{drt}}$ term s' such that $\mathrm{ACP}^+_{\mathrm{drt}} \vdash r_1' \parallel r_2' = s'$. So $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = \underline{\underline{c}} \cdot (r_1' \parallel r_2') = \underline{\underline{c}} \cdot s'$, and $\underline{\underline{c}} \cdot s'$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(q) $r_1 \equiv a \cdot r_1'$ and $r_2 \equiv \underline{\underline{b}} \cdot r_2'$ for some $a, b \in A_\delta$ and some basic terms r_1' and r_2'. This case is treated symmetrically to the previous case.

(r) $r_1 \equiv a \cdot r_1'$ and $r_2 \equiv b \cdot r_2'$ for some $a, b \in A_\delta$ and some basic terms r_1' and r_2'. Suppose that $\gamma(a,b) = c$. Then we have $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = a \cdot r_1' \mid b \cdot r_2' = c \cdot (r_1' \parallel r_2')$. By the induction hypothesis there exists a closed $\mathrm{BPA}_{\mathrm{drt}}$ term s' such that $\mathrm{ACP}^+_{\mathrm{drt}} \vdash r_1' \parallel r_2' = s'$. So $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = c \cdot (r_1' \parallel r_2') = \underline{\underline{c}} \cdot s'$, and $c \cdot s'$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(s) $r_1 \equiv r_1' + r_1''$ for some basic terms r_1' and r_1'', and r_2 is of arbitrary form. Then $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = (r_1' + r_1'') \mid r_2 = r_1' \mid r_2 + r_1'' \mid r_2$. By the induction hypothesis there exist closed $\mathrm{BPA}_{\mathrm{drt}}$ terms p_1 and p_2 such that $\mathrm{ACP}^+_{\mathrm{drt}} \vdash r_1' | r_2 = p_1$ and $\mathrm{ACP}^+_{\mathrm{drt}} \vdash r_1'' | r_2 = p_2$. So, we have $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 | t_2 = r_1' | r_2 + r_1'' | r_2 = p_1 + p_2$, and $p_1 + p_2$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(t) r_1 is of arbitrary form and $r_2 \equiv r_2' + r_2''$ for some basic terms r_2' and r_2''. This case is treated symmetrically to the previous case.

(u) $r_1 \equiv \sigma_{\mathrm{rel}}(r_1')$ and r_1' a basic term, and r_2 is of arbitrary form. Using Lemma 2.8.13 we distinguish four cases:

i. $r_2 = \dot{\delta}$. Then we have: $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \sigma_{\mathrm{rel}}(r_1') \mid \dot{\delta} = \dot{\delta}$, and $\dot{\delta}$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

ii. $r_2 = \nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}$. Then $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 | t_2 = r_1 | r_2 = \sigma_{\mathrm{rel}}(r_1') | (\nu_{\mathrm{rel}}(r_2) + \underline{\underline{\delta}}) = \underline{\underline{\delta}}$, and $\underline{\underline{\delta}}$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

iii. $r_2 = \lfloor r_2 \rfloor^{\omega}$. Then we have: $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \sigma_{\mathrm{rel}}(r_1') \mid \lfloor r_2 \rfloor^{\omega} = \sigma_{\mathrm{rel}}(r_1') \mid (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(\lfloor r_2 \rfloor^{\omega}) = \sigma_{\mathrm{rel}}(r_1' \mid \lfloor r_2 \rfloor^{\omega}) = \sigma_{\mathrm{rel}}(r_1' \mid r_2)$. By the induction hypothesis there is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term p such that $\mathrm{ACP}^+_{\mathrm{drt}} \vdash r_1' \mid r_2 = p$. But then also $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = \sigma_{\mathrm{rel}}(r_1' \mid r_2) = \sigma_{\mathrm{rel}}(p)$, and $\sigma_{\mathrm{rel}}(p)$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

iv. $r_2 = \nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')$ for a basic term r_2' such that $n(r_2') < n(r_2)$. Then we have: $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = r_1 \mid r_2 = \sigma_{\mathrm{rel}}(r_1') \mid (\nu_{\mathrm{rel}}(r_2) + \sigma_{\mathrm{rel}}(r_2')) = \sigma_{\mathrm{rel}}(r_1') \mid \sigma_{\mathrm{rel}}(r_2') = \sigma_{\mathrm{rel}}(r_1' \mid r_2')$. By the induction hypothesis there is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term p such that $\mathrm{ACP}^+_{\mathrm{drt}} \vdash r_1' \mid r_2' = p$. But then also $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mid t_2 = \sigma_{\mathrm{rel}}(r_1' \mid r_2') = \sigma_{\mathrm{rel}}(p)$, and $\sigma_{\mathrm{rel}}(p)$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

(v) r_1 is of arbitrary form and $r_2 \equiv \sigma_{\mathrm{rel}}(r_2')$ and r_2' a basic term. This case is treated symmetrically to the previous case.

(xi). $t \equiv t_1 \parallel t_2$ for closed $\mathrm{ACP}_{\mathrm{drt}}$ terms t_1 and t_2. Then $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \parallel t_2 = t_1 \mathbin{⫗} t_2 + t_2 \mathbin{⫗} t_1 + t_1 | t_2$. By (ix) and (x) there are closed $\mathrm{BPA}_{\mathrm{drt}}$ terms p_1, p_2, and p_3, such that $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \mathbin{⫗} t_2 = p_1$, $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_2 \mathbin{⫗} t_1 = p_2$, and $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 | t_2 = p_3$. But then also $\mathrm{ACP}^+_{\mathrm{drt}} \vdash t_1 \parallel t_2 = t_1 \mathbin{⫗} t_2 + t_2 \mathbin{⫗} t_1 + t_1 | t_2 = p_1 + p_2 + p_3$, and $p_1 + p_2 + p_3$ is a closed $\mathrm{BPA}_{\mathrm{drt}}$ term.

■

Corollary 3.7.12 (Elimination for $\mathrm{ACP}^+_{\mathrm{drt}}$)
Let t be a closed ACP_{drt} term. Then there is a basic term s such that $ACP^+_{drt} \vdash s = t$.

Proof This follows immediately from:

(i). The elimination theorem for $\mathrm{ACP}^+_{\mathrm{drt}}$ (see Theorem 3.7.11),

(ii). the elimination theorem for $\mathrm{BPA}^+_{\mathrm{drt}}$ (see Theorem 2.8.16),

(iii). the fact that all axioms of $\mathrm{BPA}^+_{\mathrm{drt}}$ are also contained in $\mathrm{ACP}^+_{\mathrm{drt}}$.

■

Remark 3.7.13 (Elimination for $\mathrm{ACP_{drt}}$)
Elimination for a somewhat different version of $\mathrm{ACP_{drt}}$ is also claimed (without proof) in Section 3.10 of [10].

Theorem 3.7.14 (Soundness of $\mathrm{ACP^{+}_{drt}}$)
The set of closed ACP_{drt} terms modulo bisimulation equivalence is a model of ACP^{+}_{drt}.

Proof We only prove soundness for the axioms of $\mathrm{ACP_{drt}}$ that have not been treated in earlier soundness proofs. Note that to extend these proofs to $\mathrm{ACP_{drt}}$, we have to check that the bisimulations given in previous soundness proofs respect the ID predicate (as required by transfer condition (iv.) in 2.7.7 on page 46). However, as the fact that they do can be easily checked, we will not give details.

Axiom DRTCM6ID Take the relation:

$$R = \{(\sigma_{\mathrm{rel}}(s) \mid (\nu_{\mathrm{rel}}(t) + \underline{\underline{\delta}}), \underline{\underline{\delta}}) \mid s, t \in C(\mathrm{ACP_{drt}})\}$$

We look at the transitions of both sides at the same time. Observe that there are no transitions possible on the left-hand side: $\sigma_{\mathrm{rel}}(s) \mid (\nu_{\mathrm{rel}}(t) + \underline{\underline{\delta}}) \nrightarrow$. Also for the right-hand side there are no transitions possible: $\underline{\underline{\delta}} \nrightarrow$. Finally, neither side satisfies the ID predicate: $\neg\mathrm{ID}(\sigma_{\mathrm{rel}}(s) \mid (\nu_{\mathrm{rel}}(t) + \underline{\underline{\delta}}))$ and $\neg\mathrm{ID}(\underline{\underline{\delta}})$.

Axiom DRTCM7ID Take the relation:

$$R = \{((\nu_{\mathrm{rel}}(s) + \underline{\underline{\delta}}) \mid \sigma_{\mathrm{rel}}(t), \underline{\underline{\delta}}) \mid s, t \in C(\mathrm{ACP_{drt}})\}$$

This case is treated symmetrically to the previous case.

Axiom DRTMID3 Take the relation:

$$R = \{(s \mid \dot{\delta}, \dot{\delta}) \mid s \in C(\mathrm{ACP_{drt}})\}$$

We look at the transitions of both sides at the same time. Observe that there are no transitions possible on the left-hand side: $s \mid \dot{\delta} \nrightarrow$. Also for the right-hand side there are no transitions possible: $\dot{\delta} \nrightarrow$. Finally, both sides satisfy the ID predicate: $\mathrm{ID}(s \mid \dot{\delta})$ and $\mathrm{ID}(\dot{\delta})$.

Axiom DRTMID4 Take the relation:

$$R = \{(\dot{\delta} \mid s, \dot{\delta}) \mid s \in C(\mathrm{ACP_{drt}})\}$$

This case is treated symmetrically to the previous case.

■

Remark 3.7.15 (Soundness of $\mathrm{ACP_{drt}}$)
Soundness of a somewhat different version of $\mathrm{ACP_{drt}}$ is also claimed (without proof) in Section 3.10 of [10].

Theorem 3.7.16 (Conservativity of $\mathrm{ACP^{+}_{drt}}$ with respect to $\mathrm{BPA^{+}_{drt}}$)
The equational specification ACP^{+}_{drt} is a conservative extension of the equational specification BPA^{+}_{drt}.

Proof In order to prove conservativity it is sufficient to verify that the following conditions are satisfied:

(i). Bisimulation equivalence is definable in terms of predicate and relation symbols only,

(ii). $\mathrm{BPA}^{+}_{\mathrm{drt}}$ is a complete axiomatization with respect to the bisimulation equivalence model induced by $T(\mathrm{BPA}_{\mathrm{drt}})$ (see Theorem 2.8.22),

(iii). $\mathrm{ACP}^{+}_{\mathrm{drt}}$ is a sound axiomatization with respect to the bisimulation equivalence model induced by $T(\mathrm{ACP}_{\mathrm{drt}})$ (see Theorem 3.7.14),

(iv). $T(\mathrm{ACP}_{\mathrm{drt}})$ is an operationally conservative extension of $T(\mathrm{BPA}_{\mathrm{drt}})$.

And in order for $T(\mathrm{ACP}_{\mathrm{drt}})$ indeed to be an operationally conservative extension of $T(\mathrm{BPA}_{\mathrm{drt}})$ we must verify the following conditions:

(i). $T(\mathrm{BPA}_{\mathrm{drt}})$ is a pure, well-founded term deduction system in path format,

(ii). $T(\mathrm{ACP}_{\mathrm{drt}})$ is a term deduction system in path format,

(iii). $T(\mathrm{BPA}_{\mathrm{drt}}) \oplus T(\mathrm{ACP}_{\mathrm{drt}})$ is defined.

That the above properties hold can be trivially checked from the relevant definitions. ■

Theorem 3.7.17 (Completeness of $\mathbf{ACP}^{+}_{\mathbf{drt}}$)
The equational specification ACP^{+}_{drt} is a complete axiomatization of the set of closed ACP_{drt} terms modulo bisimulation equivalence.

Proof By Verhoef's General Completeness Theorem (see [25], or Theorem 2.4.26 of [13]) this follows immediately from:

(i). $\mathrm{ACP}^{+}_{\mathrm{drt}}$ has the elimination property for $\mathrm{BPA}_{\mathrm{drt}}$ (see Theorem 3.7.11),

(ii). $\mathrm{ACP}^{+}_{\mathrm{drt}}$ is a conservative extension of $\mathrm{BPA}^{+}_{\mathrm{drt}}$ (see Theorem 3.7.16).

■

Remark 3.7.18 (Completeness of $\mathbf{ACP}_{\mathbf{drt}}$)
Completeness of a somewhat different version of $\mathrm{ACP}_{\mathrm{drt}}$ is also claimed (without proof) in Section 3.10 of [10].

Definition 3.7.19 (Axioms for the Communication Merge and Delayable Actions)
We define the Axioms USDCF1–USDCF2 and USDCM2–USDCM4 for the ultimate start delay with respect to the communication merge as shown in Table 39 on the following page. Note that they precisely correspond to the equalities of Proposition 3.7.8.

Corollary 3.7.20 (Soundness of $\mathbf{ACP}_{\mathbf{drt}}$ + USD1–7 + USDCF1–2 + USDCM2–4)
The set of closed ACP_{drt} terms modulo bisimulation equivalence is a model of ACP_{drt} + USD1–USD7 + USDCF1–USDCF2 + USDCM2–USDCM4.

$a \mid b = c$	if $\gamma(a,b) = c \neq \delta$	USDCF1
$a \mid b = \delta$	if $\gamma(a,b) = \delta$	USDCF2
$a \mid b \cdot x = (a \mid b) \cdot x$		USDCM2
$a \cdot x \mid b = (a \mid b) \cdot x$		USDCM3
$a \cdot x \mid b \cdot y = (a \mid b) \cdot (x \parallel y)$		USDCM4

Table 39: Axioms for the communication merge and delayable actions.

Proof This follows directly from the soundness of $\mathrm{ACP}^{+}_{\mathrm{drt}}$ (see Theorem 3.7.14 on page 121) and the fact that Axiom USD1–USD7, USDCF1–USDCF2, and USDCM2–USDCM4 are derivable in $\mathrm{ACP}^{+}_{\mathrm{drt}}$ (see Proposition 3.7.7 on page 110 and Proposition 3.7.8 on page 111). ■

Corollary 3.7.21 (Completeness of $\mathbf{ACP_{drt}}$ + USD1–7 + USDCF1–2 + USDCM2–4)
If we add Axioms USD1–USD4 of Table 18 on page 45, Axiom USD5 of Table 23 on page 68, Axioms USD6–USD7 of Table 36 on page 109, and Axioms USDCF1–USDCF2 and USDCM2–USDCM4 of Table 39 to ACP_{drt}, we again have a complete axiomatization of the set of closed ACP_{drt} terms modulo bisimulation equivalence.

Proof Careful inspection of the dependencies between the proofs in this section reveals that the proof of Theorem 3.7.17 only relies upon RSP(USD) to ensure Proposition 3.7.7(i)–(vii) and Proposition 3.7.8. So, we obviously do not need RSP(USD) anymore if we add the corresponding Axioms USD1–USD7, USDCF1–USDCF2, and USDCM2–USDCM4. Note that in this way we get a purely equational axiomatization (i.e. without conditional axioms or principles). Note also that Axioms CF1–CF2 and USDCF1–USDCF2 are not conditional axioms, but axiom *schemes*, as are all axioms that contain an atomic action. ■

Definition 3.7.22 (Axiom for the Ultimate Start Delay and Communication Merge)
We define the Axiom USD8 for the ultimate start delay of a communication merge as shown in Table 40.

$$\lfloor x \mid y \rfloor^{\omega} = \lfloor x \rfloor^{\omega} \mid \lfloor y \rfloor^{\omega} \qquad \text{USD8}$$

Table 40: Axiom for $\lfloor x \mid y \rfloor^{\omega}$.

Theorem 3.7.23 (Soundness of USD8)
The set of closed ACP_{drt} terms modulo bisimulation equivalence is a model of USD8.

Proof Take the relation:

$$R = \{(s,s), (\lfloor s \mid t \rfloor^{\omega}, \lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega}) \,|\, s \in C(\mathrm{ACP_{drt}})\}$$

First we look at the transitions of the left-hand side:

(i). Suppose $\lfloor s \mid t \rfloor^{\omega} \xrightarrow{a} p$. By inspection of the deduction rules we distinguish the following cases:

(a) $s \xrightarrow{b} p_1$, $t \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_1 \parallel p_2$. Then $\lfloor s \rfloor^{\omega} \xrightarrow{b} p_1$ and $\lfloor t \rfloor^{\omega} \xrightarrow{c} p_2$, so $\lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega} \xrightarrow{a} p_1 \parallel p_2$, and $(p_1 \parallel p_2, p_1 \parallel p_2) \in R$.

(b) $s \xrightarrow{b} \surd$, $t \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_2$. Then $\lfloor s \rfloor^{\omega} \xrightarrow{b} \surd$ and $\lfloor t \rfloor^{\omega} \xrightarrow{c} p_2$, so $\lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega} \xrightarrow{a} p_2$, and $(p_2, p_2) \in R$.

(c) $s \xrightarrow{b} p_1$, $t \xrightarrow{c} \surd$, $\gamma(b,c) = a$, and $p \equiv p_1$. Then $\lfloor s \rfloor^{\omega} \xrightarrow{b} p_1$ and $\lfloor t \rfloor^{\omega} \xrightarrow{c} \surd$, so $\lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega} \xrightarrow{a} p_1$, and $(p_1, p_1) \in R$.

(ii). Suppose $\lfloor s \mid t \rfloor^{\omega} \xrightarrow{a} \surd$. By inspection of the deduction rules we can conclude that $s \xrightarrow{b} \surd$, $t \xrightarrow{c} \surd$, and $\gamma(b,c) = a$. Then $\lfloor s \rfloor^{\omega} \xrightarrow{b} \surd$ and $\lfloor t \rfloor^{\omega} \xrightarrow{c} \surd$, so $\lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega} \xrightarrow{a} \surd$.

(iii). Suppose $\lfloor s \mid t \rfloor^{\omega} \xrightarrow{\sigma} p$. By inspection of the deduction rules we can conclude that $p \equiv \lfloor s \mid t \rfloor^{\omega}$. We also have $\lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega} \xrightarrow{\sigma} \lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega}$, and $(\lfloor s \mid t \rfloor^{\omega}, \lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega}) \in R$.

Secondly, we look at the transition of the right-hand side:

(i). Suppose $\lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega} \xrightarrow{a} p$. By inspection of the deduction rules we distinguish the following cases:

(a) $s \xrightarrow{b} p_1$, $t \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_1 \parallel p_2$. Then $\lfloor s \mid t \rfloor^{\omega} \xrightarrow{c} p_1 \parallel p_2$, and $(p_1 \parallel p_2, p_1 \parallel p_2) \in R$.

(b) $s \xrightarrow{b} \surd$, $t \xrightarrow{c} p_2$, $\gamma(b,c) = a$, and $p \equiv p_2$. Then $\lfloor s \mid t \rfloor^{\omega} \xrightarrow{a} p_2$, and $(p_2, p_2) \in R$.

(c) $s \xrightarrow{b} p_1$, $t \xrightarrow{c} \surd$, $\gamma(b,c) = a$, and $p \equiv p_1$. Then $\lfloor s \mid t \rfloor^{\omega} \xrightarrow{a} p_1$, and $(p_1, p_1) \in R$.

(ii). Suppose $\lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega} \xrightarrow{a} \surd$. By inspection of the deduction rules we can conclude that $s \xrightarrow{b} \surd$, $t \xrightarrow{c} \surd$, and $\gamma(b,c) = a$. Then $\lfloor s \mid t \rfloor^{\omega} \xrightarrow{a} \surd$, and we are done.

(iii). Suppose $\lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega} \xrightarrow{\sigma} p$. By inspection of the deduction rules we can conclude that $p \equiv \lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega}$. We also have $\lfloor s \mid t \rfloor^{\omega} \xrightarrow{\sigma} \lfloor s \mid t \rfloor^{\omega}$, and $(\lfloor s \mid t \rfloor^{\omega}, \lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega}) \in R$.

Finally, we look at the immediate deadlock predicate. Neither side has immediate deadlock: $\neg \mathrm{ID}(\lfloor s \mid t \rfloor^{\omega})$ and $\neg \mathrm{ID}(\lfloor s \rfloor^{\omega} \mid \lfloor t \rfloor^{\omega})$ (note that ultimate start delay removes immediate deadlock, see Remark 2.8.5 on page 54). ■

Proposition 3.7.24 (Properties of $\mathbf{ACP_{drt}}$, Part IV)
For ACP_{drt} terms x and y and any $a, b, c \in A_{\delta}$, we have the following equalities:

(i). $ACP_{drt} + USD8 \vdash a \mid b = c \qquad \text{if } \gamma(a,b) = c \neq \delta$

(ii). $ACP_{drt} + USD8 \vdash a \mid b = \delta \qquad \text{if } \gamma(a,b) = \delta$

(iii). $ACP_{drt} + USD2 + USD8 \vdash a \mid b \cdot x = (a \mid b) \cdot x$

(iv). $ACP_{drt} + USD2 + USD8 \vdash a \cdot x \mid b = (a \mid b) \cdot x$

(v). $ACP_{drt} + USD2 + USD8 \vdash a \cdot x \mid b \cdot y = (a \mid b) \cdot (x \parallel y)$

Proof

(i). $\mathrm{ACP_{drt}} + \mathrm{USD8} \vdash a \mid b = \lfloor \underline{\underline{a}} \rfloor^{\omega} \mid \lfloor \underline{\underline{b}} \rfloor^{\omega} = \lfloor \underline{\underline{a}} \mid \underline{\underline{b}} \rfloor^{\omega} = \lfloor \underline{\underline{c}} \rfloor^{\omega} = c$

(ii). $\mathrm{ACP_{drt}} + \mathrm{USD8} \vdash a \mid b = \lfloor \underline{\underline{a}} \rfloor^{\omega} \mid \lfloor \underline{\underline{b}} \rfloor^{\omega} = \lfloor \underline{\underline{a}} \mid \underline{\underline{b}} \rfloor^{\omega} = \lfloor \underline{\underline{\delta}} \rfloor^{\omega} = \delta$

(iii). $\mathrm{ACP_{drt}} + \mathrm{USD2} + \mathrm{USD8} \vdash a \mid b \cdot x = \lfloor \underline{\underline{a}} \rfloor^{\omega} \mid \lfloor \underline{\underline{b}} \rfloor^{\omega} \cdot x = \lfloor \underline{\underline{a}} \rfloor^{\omega} \mid \lfloor \underline{\underline{b}} \cdot x \rfloor^{\omega} = \lfloor \underline{\underline{a}} \mid \underline{\underline{b}} \cdot x \rfloor^{\omega} = \lfloor (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot x \rfloor^{\omega} = \lfloor (\underline{\underline{a}} \mid \underline{\underline{b}}) \rfloor^{\omega} \cdot x = (\lfloor \underline{\underline{a}} \rfloor^{\omega} \mid \lfloor \underline{\underline{b}} \rfloor^{\omega}) \cdot x = (a \mid b) \cdot x$

(iv). $\mathrm{ACP_{drt}} + \mathrm{USD2} + \mathrm{USD8} \vdash a \cdot x \mid b = \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot x \mid \lfloor \underline{\underline{b}} \rfloor^{\omega} = \lfloor \underline{\underline{a}} \cdot x \rfloor^{\omega} \mid \lfloor \underline{\underline{b}} \rfloor^{\omega} = \lfloor \underline{\underline{a}} \cdot x \mid \underline{\underline{b}} \rfloor^{\omega} = \lfloor (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot x \rfloor^{\omega} = \lfloor (\underline{\underline{a}} \mid \underline{\underline{b}}) \rfloor^{\omega} \cdot x = (\lfloor \underline{\underline{a}} \rfloor^{\omega} \mid \lfloor \underline{\underline{b}} \rfloor^{\omega}) \cdot x = (a \mid b) \cdot x$

(v). $\mathrm{ACP_{drt}} + \mathrm{USD2} + \mathrm{USD8} \vdash a \cdot x \mid b \cdot y = \lfloor \underline{\underline{a}} \rfloor^{\omega} \cdot x \mid \lfloor \underline{\underline{b}} \rfloor^{\omega} \cdot y = \lfloor \underline{\underline{a}} \cdot x \rfloor^{\omega} \mid \lfloor \underline{\underline{b}} \cdot y \rfloor^{\omega} = \lfloor \underline{\underline{a}} \cdot x \mid \underline{\underline{b}} \cdot y \rfloor^{\omega} = \lfloor (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot (x \parallel y) \rfloor^{\omega} = \lfloor (\underline{\underline{a}} \mid \underline{\underline{b}}) \rfloor^{\omega} \cdot (x \parallel y) = (\lfloor \underline{\underline{a}} \rfloor^{\omega} \mid \lfloor \underline{\underline{b}} \rfloor^{\omega}) \cdot (x \parallel y) = (a \mid b) \cdot (x \parallel y)$

■

Remark 3.7.25 (Properties of ACP$_{\mathrm{drt}}$, Part IV)
Note that the equalities of Proposition 3.7.24 on the preceding page correspond precisely to the equalities of Proposition 3.7.8 on page 111 and Axioms USDCF1–USDCF2 and USDCM2–USDCM4 from Definition 3.7.19 on page 122.

Corollary 3.7.26 (Soundness of ACP$_{\mathrm{drt}}$ + USD1–USD8)
The set of closed ACP$_{drt}$ terms modulo bisimulation equivalence is a model of ACP$_{drt}$ + USD1–USD8.

Proof This follows directly from the soundness of ACP$^{+}_{\mathrm{drt}}$ (see Theorem 3.7.14 on page 121) and the facts that Axioms USD1–USD7 are derivable in ACP$^{+}_{\mathrm{drt}}$ (see Proposition 3.7.7 on page 110) and that Axiom USD8 is sound (see Theorem 3.7.23 on page 123). ■

Corollary 3.7.27 (Completeness of ACP$_{\mathrm{drt}}$ + USD1–USD8)
If we add Axioms USD1–USD4 of Table 18 on page 45, Axiom USD5 of Table 23 on page 68, Axioms USD6–USD7 of Table 36 on page 109 and Axiom USD8 of Table 40 on page 123 to ACP$_{drt}$, we again have a complete axiomatization of the set of closed ACP$_{drt}$ terms modulo bisimulation equivalence.

Proof As show in Proposition 3.7.24, if we have the add Axioms USD2 and USD8 to the axioms of ACP$_{\mathrm{drt}}$, we can (for closed terms) derive Axiom USDCF1–USDCF2 and USDCM2–USDCM4. Using Corollary 3.7.21 on page 123, we can now trivially derive the desired result. ■

4 Conclusions

To begin with, we are reasonably confident that the axiomatizations listed in this paper are sound and complete. An overview of the main theorems is listed in Table 44 on page 136.

Before we started this paper, we were also confident of the soundness and completeness of these axiomatizations, but at that time wrongly so. We discovered that the axiomatizations we started out with, most of which have been published and claimed sound and complete before, were neither sound nor complete. We highlight two characteristic cases:

- Weakening DRT4 to DRT4A brings the need to introduce DRT5, something we did not realize at first. This left all interesting theories incomplete. Only when DRT5 was needed in the proof of Lemma 2.6.16(iv), we found out about this mistake.
- Introducing $\dot{\delta}$ in a context that supports communication brings the need to weaken DRTCM6 to DRTCM6ID, which we did not realize due to the "intuitive" and "obvious" nature of DRTCM6. This left some theories unsound. We found out this problem after we could not complete the last few "trivial details" of the proof of Theorem 3.7.14.

Both these problems were discovered when we were writing out all details of "trivial" proofs, proofs which we had originally not planned to do at all. So, we eventually decided to give as much and as detailed proofs as reasonably manageable. And, as to be expected, we found some more mistakes like the ones listed above. As a side-result, we gained insight into the various aspects of axiomatizations.

Firstly, when we weigh the merits of the "$\nu_{\mathrm{rel}}/\sigma_{\mathrm{rel}}$ axiomatization style" of [11] (with theories like $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID and $\mathrm{ACP}^{-}_{\mathrm{drt}}$-ID) against those of the "classic axiomatization style" of [10] (with theories like $\mathrm{PA}^{-}_{\mathrm{drt}}$-ID$'$and $\mathrm{ACP}^{-}_{\mathrm{drt}}$-ID$'$), we conclude that the $\nu_{\mathrm{rel}}/\sigma_{\mathrm{rel}}$ style is better suited towards practical applications, as it makes calculations easier. However, from a theoretical viewpoint it's troublesome: it does not lend itself well to term rewriting system analysis, and worse, it does not seem to be compatible with the addition of the empty step, as is shown in [12]. On the other hand, the classic style is not ideal either. It appears less intuitive, and it needs more axioms. Compare for example Axioms DRTM5–DRTM6 of Table 25 on page 70 with Axioms DRTM7–DRTM11 of Table 27 on page 76. Here the classic style needs five axioms to do what the $\nu_{\mathrm{rel}}/\sigma_{\mathrm{rel}}$ style can do much clearer in two axioms. Consequently, calculations in the classic style are much longer too.

Second, we have shown how to eliminate the recursion principle RSP(USD) from the theories that contain it. As shown in Corollaries 2.6.22, 2.8.26, 3.6.18, 3.7.21, and 3.7.27, one can straightforwardly derive unconditional axioms to replace the conditional principle RSP(USD). The recipe is always the same: identify in the correctness proof of the conditional theory the places where RSP(USD) is used, put those applications in a separate lemma, and introduce an axiom for every clause of that lemma. Using this recipe, we introduced Axioms USD1–USD8, USDCF1–USDCF2, and USDCM2–USDCM4. The advantage is clear: having a fully unconditional theory enables us to reason fully algebraically, giving us a fuller apparatus of methods to work with. On the other hand, the principle RSP(USD) is clean, neat, and simple, and can be applied in any theory, while the "USD axiomatization style" requires new axioms for every new theory. That this can lead to unwieldy theories can be observed from [10]. We feel that the “RSP(USD) axiomatization style”, which till now has only appeared in the rather obscure papers [9, 11], deserves a wider audience.

Third, we find that the absence of the *empty step* (a constant $\underline{\underline{\varepsilon}}$ such that $\underline{\underline{\varepsilon}} \cdot x = x = x \cdot \underline{\underline{\varepsilon}}$, see [12] for details) is a major nuisance. To begin with, the empty step would allow us to express our axioms much more compact. For example, Axiom DRTM2 would be just an instance of Axiom DRTM3 if the x in the latter could take the value $\underline{\underline{\varepsilon}}$. Similarly, Axioms DRTCM2–DRTCM4 could also be collapsed to just one axiom with the help of $\underline{\underline{\varepsilon}}$ (and see [7] for a theory in which it is hard to find *any* axiom that could not be formulated better with the help of the empty step!). The absence of $\underline{\underline{\varepsilon}}$ is even felt worse when doing calculations. If we look for example at the proof of Theorem 3.7.11, we see we have to distinguish 49 cases(!) when doing simultaneous induction on two variables, as a basic term in $\mathrm{ACP}_{\mathrm{drt}}$ can take seven essentially different forms. With the help of $\underline{\underline{\varepsilon}}$, we could reduce this to five forms, and only 25 cases would have to be considered when doing induction on two variables. Similar considerations hold for the proof of Theorem 3.4.12 on page 88, where the absence of $\underline{\underline{\varepsilon}}$ for some case even forces us into a *sixteen-fold* (sic!) increase in proof obligations. We conclude that there is a clear need for the empty step in discrete-time process algebra.

Then, we hope that this paper can serve as a reference point: to our knowledge this is the first paper that *extensively* lists all important discrete-time process algebra theories, together with all relevant definitions and elementary theorems. Furthermore, as all proofs in this paper are constructive, it should now be easy to develop a tool that can automatically rewrite two bisimilar $\mathrm{ACP}_{\mathrm{drt}}$ terms into one another.

Finally, note that we have surveyed several distinct methods for proving soundness and completeness. To prove soundness we have used:

- the direct method (see Remark 2.4.13 on page 17),
- the indirect method (see Remark 2.6.20 on page 44), and,
- the ground equivalence method (see Remark 3.3.14 on page 82).

To prove completeness we have used:

- the direct method (see Remark 2.2.18 on page 9),
- the indirect method (see Remark 2.6.20 on page 44),
- Verhoef’s method (see Remark 3.2.14 on page 75), and,
- the ground equivalence method (see Remark 3.3.14 on page 82).

We believe that this spectrum of methods provides a convenient starting point to prove soundness and completeness of most (timed) process algebras that have been described in the literature, with the exception of theories that support abstraction.

As far as future research is concerned: we would like to generalize our results to a setting that includes abstraction. This seems however not at all trivial, and may require a substantial effort. Furthermore, as noted above, further research on the empty step is justified. This work is currently in progress, and will be published as [12].

5 Acknowledgments

Note that some of the results presented here were taken from [13]: our Lemma 2.2.19 is their Lemma 2.2.34 (but with a more detailed proof), our Theorem 2.2.22 is their Theorem 2.2.35 (with a more detailed proof), our Theorem 2.3.11 is their Theorem 2.5.5, and finally our Theorem 2.4.14 is their Theorem 2.12.4 (also with a more detailed proof).

We like to thank Jos Baeten, Roel Bloo, Sjouke Mauw, and Hans Mulder for their various useful suggestions.

A Nomenclature and notational issues

In this section we present a quick overview of the nomenclature used in ACP-style discrete-time process algebras.

A.1 Signatures

In Table 41 we give the signature of some process algebras, among which all those treated in this paper.

	$+$	$\cdot$	a	δ	$\lfloor\ \rfloor^{\omega}$	σ_{rel}	ν_{rel}	$\underline{\underline{a}}$	$\underline{\underline{\delta}}$	$\dot{\delta}$	$\parallel$	$\lfloor\!\lfloor$	$\mid$
BPA	•	•	•										
BPA_{δ}	•	•	•	•									
PA	•	•	•								•	•	
PA_{δ}	•	•	•	•							•	•	
ACP	•	•	•	•							•	•	•
$\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-}\delta$	•	•				•		•					
$\mathrm{BPA}^{-}_{\mathrm{drt}}\text{-ID}$	•	•				•	•	•	•				
$\mathrm{BPA}^{-}_{\mathrm{drt}}$	•	•				•	•	•	•	•			
$\mathrm{BPA}_{\mathrm{drt}}\text{-ID}$	•	•	•	•	•	•	•	•	•				
$\mathrm{BPA}_{\mathrm{drt}}$	•	•	•	•	•	•	•	•	•	•			
$\mathrm{PA}^{-}_{\mathrm{drt}}\text{-ID}$	•	•				•	•	•	•		•	•	
$\mathrm{PA}^{-}_{\mathrm{drt}}$	•	•				•	•	•	•	•	•	•	
$\mathrm{PA}_{\mathrm{drt}}\text{-ID}$	•	•	•	•	•	•	•	•	•		•	•	
$\mathrm{PA}_{\mathrm{drt}}$	•	•	•	•	•	•	•	•	•	•	•	•	
$\mathrm{ACP}^{-}_{\mathrm{drt}}\text{-ID}$	•	•				•	•	•	•		•	•	•
$\mathrm{ACP}^{-}_{\mathrm{drt}}$	•	•				•	•	•	•	•	•	•	•
$\mathrm{ACP}_{\mathrm{drt}}\text{-ID}$	•	•	•	•	•	•	•	•	•		•	•	•
$\mathrm{ACP}_{\mathrm{drt}}$	•	•	•	•	•	•	•	•	•	•	•	•	•

Table 41: Signatures of some process algebras.

In naming discrete-time process algebras we have, among others, the following conventions:

- The subscript "$_{\mathrm{drt}}$" signifies "discrete relative time", so we have the relative-time time-unit delay ("σ_{rel}"), and furthermore either non-delayable actions (double underlined: "$\underline{\underline{a}}$"), or delayable actions (no special notation: "a"), or both. If we have the undelayable deadlock constant ("$\underline{\underline{\delta}}$"), we also have the "now" operator ("ν_{rel}"), and vice versa.

- The superscript "$^{-}$" signifies that we do not have delayable actions ("a"), so only the non-delayable actions remain ("$\underline{\underline{a}}$").

- The postfix "$-\delta$" signifies the absence of all deadlock constants: non-delayable ("$\underline{\underline{\delta}}$"), delayable ("$\delta$"), or immediate ("$\dot{\delta}$").

- The postfix "–ID" signifies the absence of the immediate deadlock constant ("$\dot{\delta}$").

A.2 Comparison with other notations

In the (older) literature about ACP-style discrete-time process algebras we sometimes find slightly different notations. Below we list the most important differences:

- We find the notation "$\mathrm{BPA}_{\mathrm{dt}}$" for "$\mathrm{BPA}^{-}_{\mathrm{drt}}$–$\delta$", "$\mathrm{BPA}_{\delta\mathrm{dt}}$" for "$\mathrm{BPA}^{-}_{\mathrm{drt}}$–ID", and "$\mathrm{PA}_{\delta\mathrm{dt}}$" for "$\mathrm{PA}^{-}_{\mathrm{drt}}$–ID" [13].

- We find the notation "$\mathrm{ACP}_{\mathrm{dt}}$" for "$\mathrm{ACP}^{-}_{\mathrm{drt}}$–ID" [6, 13, 23],

- We find the notations "σ_d" and "$\sigma_{\mathrm{rel}(1)}$" for "σ_{rel}" [6, 13, 23].

- We find the notation "$\mathrm{cts}(a)$" ("*c*urrent *t*ime *s*lice") for the undelayable action "$\underline{\underline{a}}$", and "$\mathrm{ats}(a)$" ("*a*ny *t*ime *s*lice") for the delayable action "a" [8, 10].

- We find the notation "$\underline{a}[1]$" for the undelayable action "$\underline{\underline{a}}$", and "$\underline{a}$" for the delayable action "a" [5, 6].

- It has been suggested to refer to the delayable deadlock constant "δ" as *livelock*, and to the immediate deadlock constant "$\dot{\delta}$" as the *immediate time stop*, *full time stop*, or *catastrophic deadlock* [4, 10].

- The unbounded start delay operator "$\lfloor\ \rfloor^{\omega}$" was first described by Nicollin and Sifakis, and is therefore also known as one of the *Nicollin-Sifakis operators* [20].

- The naming scheme for axioms (A1, A2, etc.) has become quite problematic. Naming is generally not applied consistently *within* ACP articles published, let alone *across* them. We have no illusions this article is any different, although, and because, we sort of tried to adhere to the latest fashions.

- Often the convention is used that the variables x, y, and z refer to open process terms, and the variables s, t, and u to closed process terms. Although some parts of this article conform to this convention, we shamefully admit that in large parts it is blatantly violated.

B Overview

B.1 Axioms

The list below shows all axioms mentioned in this paper, in alphabetical order. Behind the name of the axioms the number is listed of the page where the axiom is introduced.

$x + y = y + x$	A1	(page 7)
$(x + y) + z = x + (y + z)$	A2	(page 7)
$x + x = x$	A3	(page 7)
$(x + y) \cdot z = x \cdot z + y \cdot z$	A4	(page 7)
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	A5	(page 7)
$x + \delta = x$	A6	(page 12)
$a + \delta = a$	A6A	(page 12)
$x + \dot{\delta} = x$	A6ID	(page 46)
$\delta \cdot x = \delta$	A7	(page 12)
$\dot{\delta} \cdot x = \dot{\delta}$	A7ID	(page 46)
$a = \lfloor \underline{\underline{a}} \rfloor^{\omega}$	ATS	(page 32)
$\nu_{\mathrm{rel}}(\underline{\underline{a}}) = \underline{\underline{a}}$	DCS1	(page 23)
$\nu_{\mathrm{rel}}(x + y) = \nu_{\mathrm{rel}}(x) + \nu_{\mathrm{rel}}(y)$	DCS2	(page 23)
$\nu_{\mathrm{rel}}(x \cdot y) = \nu_{\mathrm{rel}}(x) \cdot y$	DCS3	(page 23)
$\nu_{\mathrm{rel}}(\sigma_{\mathrm{rel}}(x)) = \underline{\underline{\delta}}$	DCS4	(page 23)
$\nu_{\mathrm{rel}}(\dot{\delta}) = \dot{\delta}$	DCSID	(page 46)
$\sigma_{\mathrm{rel}}(x) + \sigma_{\mathrm{rel}}(y) = \sigma_{\mathrm{rel}}(x + y)$	DRT1	(page 14)
$\sigma_{\mathrm{rel}}(x) \cdot y = \sigma_{\mathrm{rel}}(x \cdot y)$	DRT2	(page 14)
$\underline{\underline{\delta}} \cdot x = \underline{\underline{\delta}}$	DRT3	(page 23)
$\underline{\underline{a}} + \underline{\underline{\delta}} = \underline{\underline{a}}$	DRT4	(page 23)
$x + \underline{\underline{\delta}} = x$	DRT4A	(page 23)
$\sigma_{\mathrm{rel}}(x) + \underline{\underline{\delta}} = \sigma_{\mathrm{rel}}(x)$	DRT5	(page 23)
$\underline{\underline{a}} \mid \underline{\underline{b}} = \underline{\underline{c}} \quad \text{if } \gamma(a,b) = c \neq \delta$	DRTCF1	(page 84)
$\underline{\underline{a}} \mid \underline{\underline{b}} = \underline{\underline{\delta}} \quad \text{if } \gamma(a,b)\,\delta$	DRTCF2	(page 84)
$x \parallel y = x \mathbin{\lfloor\!\lfloor} y + y \mathbin{\lfloor\!\lfloor} x + x \mid y$	DRTCM1	(page 84)
$\underline{\underline{a}} \mid \underline{\underline{b}} \cdot x = (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot x$	DRTCM2	(page 84)
$\underline{\underline{a}} \cdot x \mid \underline{\underline{b}} = (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot x$	DRTCM3	(page 84)
$\underline{\underline{a}} \cdot x \mid \underline{\underline{b}} \cdot y = (\underline{\underline{a}} \mid \underline{\underline{b}}) \cdot (x \parallel y)$	DRTCM4	(page 84)
$\sigma_{\mathrm{rel}}(x) \mid \sigma_{\mathrm{rel}}(y) = \sigma_{\mathrm{rel}}(x \mid y)$	DRTCM5	(page 84)
$\sigma_{\mathrm{rel}}(x) \mid \nu_{\mathrm{rel}}(y) = \underline{\underline{\delta}}$	DRTCM6	(page 85)
$\sigma_{\mathrm{rel}}(x) \mid (\nu_{\mathrm{rel}}(y) + \underline{\underline{\delta}}) = \underline{\underline{\delta}}$	DRTCM6ID	(page 110)
$\nu_{\mathrm{rel}}(x) \mid \sigma_{\mathrm{rel}}(y) = \underline{\underline{\delta}}$	DRTCM7	(page 85)
$(\nu_{\mathrm{rel}}(x) + \underline{\underline{\delta}}) \mid \sigma_{\mathrm{rel}}(y) = \underline{\underline{\delta}}$	DRTCM7ID	(page 110)
$\underline{\underline{a}} \mid \sigma_{\mathrm{rel}}(x) = \underline{\underline{\delta}}$	DRTCM8	(page 94)

$\sigma_{\mathrm{rel}}(x) \mid \underline{\underline{a}} = \underline{\underline{\delta}}$	DRTCM9	(page 94)
$\underline{\underline{a}} \cdot x \mid \sigma_{\mathrm{rel}}(y) = \underline{\underline{\delta}}$	DRTCM10	(page 94)
$\sigma_{\mathrm{rel}}(x) \mid \underline{\underline{a}} \cdot y = \underline{\underline{\delta}}$	DRTCM11	(page 94)
$(x + y) \mid z = x \mid z + y \mid z$	DRTCM12	(page 84)
$x \mid (y + z) = x \mid y + x \mid z$	DRTCM13	(page 84)
$x \parallel y = x \mathbin{\lfloor\!\!\lfloor} y + y \mathbin{\lfloor\!\!\lfloor} x$	DRTM1	(page 69)
$\underline{\underline{a}} \mathbin{\lfloor\!\!\lfloor} x = \underline{\underline{a}} \cdot x$	DRTM2	(page 69)
$\underline{\underline{a}} \mathbin{\lfloor\!\!\lfloor} (x + \underline{\underline{\delta}}) = \underline{\underline{a}} \cdot (x + \underline{\underline{\delta}})$	DRTM2ID	(page 100)
$\underline{\underline{a}} \cdot x \mathbin{\lfloor\!\!\lfloor} y = \underline{\underline{a}} \cdot (x \parallel y)$	DRTM3	(page 69)
$\underline{\underline{a}} \cdot x \mathbin{\lfloor\!\!\lfloor} (y + \underline{\underline{\delta}}) = \underline{\underline{a}} \cdot (x \parallel (y + \underline{\underline{\delta}}))$	DRTM3ID	(page 100)
$(x + y) \mathbin{\lfloor\!\!\lfloor} z = x \mathbin{\lfloor\!\!\lfloor} z + y \mathbin{\lfloor\!\!\lfloor} z$	DRTM4	(page 69)
$\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\!\lfloor} \nu_{\mathrm{rel}}(y) = \underline{\underline{\delta}}$	DRTM5	(page 70)
$\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\!\lfloor} (\nu_{\mathrm{rel}}(y) + \underline{\underline{\delta}}) = \underline{\underline{\delta}}$	DRTM5ID	(page 100)
$\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\!\lfloor} (\nu_{\mathrm{rel}}(y) + \sigma_{\mathrm{rel}}(z)) = \sigma_{\mathrm{rel}}(x \mathbin{\lfloor\!\!\lfloor} z)$	DRTM6	(page 70)
$\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\!\lfloor} \underline{\underline{a}} = \underline{\underline{\delta}}$	DRTM7	(page 76)
$\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\!\lfloor} \underline{\underline{a}} \cdot y = \underline{\underline{\delta}}$	DRTM8	(page 76)
$\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\!\lfloor} (\underline{\underline{a}} + y) = \sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\!\lfloor} y$	DRTM9	(page 76)
$\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\!\lfloor} (\underline{\underline{a}} \cdot y + z) = \sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\!\lfloor} z$	DRTM10	(page 76)
$\sigma_{\mathrm{rel}}(x) \mathbin{\lfloor\!\!\lfloor} \sigma_{\mathrm{rel}}(y) = \sigma_{\mathrm{rel}}(x \mathbin{\lfloor\!\!\lfloor} y)$	DRTM11	(page 76)
$x \mathbin{\lfloor\!\!\lfloor} \dot{\delta} = \dot{\delta}$	DRTMID1	(page 100)
$\dot{\delta} \mathbin{\lfloor\!\!\lfloor} x = \dot{\delta}$	DRTMID2	(page 100)
$x \mid \dot{\delta} = \dot{\delta}$	DRTMID3	(page 110)
$\dot{\delta} \mid x = \dot{\delta}$	DRTMID4	(page 110)
$\sigma_{\mathrm{rel}}(\dot{\delta}) = \underline{\underline{\delta}}$	DRTSID	(page 46)
$\lfloor x \rfloor^{\omega} = \nu_{\mathrm{rel}}(x) + \sigma_{\mathrm{rel}}(\lfloor x \rfloor^{\omega})$	USD	(page 32)
$\lfloor a \rfloor^{\omega} = a$	USD1	(page 45)
$\lfloor x \cdot y \rfloor^{\omega} = \lfloor x \rfloor^{\omega} \cdot y$	USD2	(page 45)
$\lfloor x + y \rfloor^{\omega} = \lfloor x \rfloor^{\omega} + \lfloor y \rfloor^{\omega}$	USD3	(page 45)
$\lfloor \sigma_{\mathrm{rel}}(x) \rfloor^{\omega} = \delta$	USD4	(page 45)
$\lfloor \dot{\delta} \rfloor^{\omega} = \delta$	USD5	(page 68)
$a \mathbin{\lfloor\!\!\lfloor} \lfloor x \rfloor^{\omega} = a \cdot \lfloor x \rfloor^{\omega}$	USD6	(page 109)
$a \cdot x \mathbin{\lfloor\!\!\lfloor} \lfloor y \rfloor^{\omega} = a \cdot (x \parallel \lfloor y \rfloor^{\omega})$	USD7	(page 109)
$\lfloor x \mid y \rfloor^{\omega} = \lfloor x \rfloor^{\omega} \mid \lfloor y \rfloor^{\omega}$	USD8	(page 123)
$a \mid b = c \quad \text{if } \gamma(a, b) = c \neq \delta$	USDCF1	(page 123)
$a \mid b = \delta \quad \text{if } \gamma(a, b) = \delta$	USDCF2	(page 123)
$a \mid b \cdot x = (a \mid b) \cdot x$	USDCM2	(page 123)
$a \cdot x \mid b = (a \mid b) \cdot x$	USDCM3	(page 123)
$a \cdot x \mid b \cdot y = (a \mid b) \cdot (x \parallel y)$	USDCM4	(page 123)

B.2 Theories

The list below gives for every theory we defined, the axioms it contains.

- BPA = A1–A5.
- BPA_δ = A1–A7.
- BPA^-_{drt}-δ = A1–A5 + DRT1–DRT2.
- BPA^-_{drt}-ID = A1–A5 + DRT1–DRT5 + DCS1–DCS4.
- BPA_{drt}-ID = A1–A5 + DRT1–DRT5 + DCS1–DCS4 + ATS + USD.
- BPA^-_{drt} = A1–A5 + A6ID + A7ID + DRT1–DRT5 + DRTSID + DCS1–DCS4 + DCSID.
- BPA_{drt} = A1–A5 + A6ID + A7ID + DRT1–DRT5 + DRTSID + DCS1–DCS4 + DCSID + ATS + USD.
- PA^-_{drt}-ID = A1–A5 + DRT1–DRT5 + DCS1–DCS4 + DRTM1–DRTM6.
- PA^-_{drt}-ID′ = A1–A5 + DRT1–DRT5 + DCS1–DCS4 + DRTM1–DRTM4 + DRTM7–DRTM11.
- ACP^-_{drt}-ID = A1–A5 + DRT1–DRT5 + DCS1–DCS4 + DRTM2–DRTM6 + DRTCM1–DRTCM7 + DRTCM12–DRTCM13 + DRTCF1–DRTCF2.
- ACP^-_{drt}-ID′ = A1–A5 + DRT1–DRT5 + DCS1–DCS4 + DRTM2–DRTM4, DRTM7–DRTM11 + DRTCM1–DRTCM5 + DRTCM8–DRTCM13 + DRTCF1–DRTCF2.
- PA_{drt} = A1–A5 + A6ID + A7ID + DRT1–DRT5 + DRTSID + DCS1–DCS4 + DCSID + ATS + USD + DRTM1 + DRTM2ID–DRTM3ID + DRTM4 + DRTM5ID + DRTM6 + DRTMID1–DRTMID2.
- ACP_{drt} = A1–A5 + A6ID + A7ID + DRT1–DRT5 + DRTSID + DCS1–DCS4 + DCSID + ATS + USD + DRTM2ID–DRTM3ID + DRTM4 + DRTM5ID + DRTM6 + DRTCM1–DRTCM5 + DRTCM6ID–DRTCM7ID + DRTCM12–DRTCM13 + DRTCF1–DRTCF2 + DRTMID1–DRTMID4.

In Table 42 on the next page we give an overview of the axioms of the (basic) process algebras treated in Section 2, for the purpose of comparing the respective theories with each other. We have the following legend:

- A "•" indicates that the axiom is present in the theory,
- A "+" indicates that the axiom is not present but can be derived (for closed terms) from the other axioms in the theory,
- A "−" indicates that the axiom does not hold in the theory,
- A "×" indicates that the axiom is meaningless in the theory, as there is a signature conflict.

In Table 43 on page 135 we give an overview of the axioms pertaining to merge operators of the (concurrent) process algebras treated in Section 3. The legend is the same as for the previous table.

	BPA	BPA_δ	$\mathrm{BPA}^{-}_{\mathrm{drt}}$-$\delta$	$\mathrm{BPA}^{-}_{\mathrm{drt}}$-ID	$\mathrm{BPA}^{+}_{\mathrm{drt}}$-ID	$\mathrm{BPA}^{-}_{\mathrm{drt}}$	$\mathrm{BPA}^{+}_{\mathrm{drt}}$
Main Axioms							
A1–5	•	•	•	•	•	•	•
DRT1–2	×	×	•	•	•	•	•
DRT3–5	×	×	×	•	•	•	•
DCS1–4	×	×	×	•	•	•	•
ATS, USD	×	×	×	×	•	×	•
ID Axioms							
A6ID–7ID	×	×	×	×	×	•	•
DCSID, DRTSID	×	×	×	×	×	•	•
Auxiliary Axioms							
A6	×	•	×	×	+	×	−
A7	×	•	×	×	+	×	+
A6A	×	+	×	×	+	×	+
DRT4A	×	×	×	+	+	−	−
USD-Style Axioms							
USD1–4	×	×	×	×	+	×	+
USD5	×	×	×	×	×	×	+

Table 42: Overview of axioms of basic process algebras.

	PA^-_{drt}–ID	PA^-_{drt}–ID′	ACP^-_{drt}–ID	ACP^-_{drt}–ID′	PA_{drt}	ACP_{drt}
Main Axioms for $\parallel$						
DRTM1	•	•	−	−	•	−
DRTM2–M3	•	•	•	•	−	−
DRTM4	•	•	•	•	•	•
DRTM5	•	+	•	+	−	−
DRTM6	•	+	•	+	•	•
DRTM7–M11	+	•	+	•	+	+
Main Axioms for $\mid$						
DRTCF1–CF2	×	×	•	•	×	•
DRTCM1–CM5	×	×	•	•	×	•
DRTCM6–CM7	×	×	•	+	×	−
DRTCM8–CM11	×	×	+	•	×	+
DRTCM12–CM13	×	×	•	•	×	•
ID Axioms						
DRTM2ID–M3ID	+	+	+	+	•	•
DRTM5ID	+	+	+	+	•	•
DRTCM6ID–CM7ID	×	×	+	+	×	•
DRTMID1–MID2	×	×	×	×	•	•
DRTMID3–MID4	×	×	×	×	×	•
USD-Style Axioms						
USD6–7	×	×	×	×	+	+
USD8	×	×	×	×	×	+
USDCF1–CF2	×	×	×	×	×	+
USDCM2–CM4	×	×	×	×	×	+

Table 43: Overview of axioms of concurrent process algebras.

B.3 Theorems

In Table 44 we give an overview of the definitions of the axioms and semantics of the process algebras listed in this paper, with corresponding theorems regarding elimination, soundness, and completeness.

	Axioms	Semantics	Elimination	Soundness	Completeness
Basic Process Algebras					
BPA	Def. 2.2.6	Def. 2.2.8	Prop. 2.2.16	Thm. 2.2.17	Thm. 2.2.22
BPA_δ	Def. 2.3.2	Def. 2.3.4	Prop. 2.3.9	Thm. 2.3.10	Thm. 2.3.11
$\mathrm{BPA}^-_{\mathrm{drt}}\text{-}\delta$	Def. 2.4.2	Def. 2.4.4	Thm. 2.4.11	Thm. 2.4.14	Thm. 2.4.17
$\mathrm{BPA}^-_{\mathrm{drt}}\text{-ID}$	Def. 2.5.2	Def. 2.5.4	Thm. 2.5.12	Thm. 2.5.14	Thm. 2.5.17
$\mathrm{BPA}^+_{\mathrm{drt}}\text{-ID}$	Def. 2.6.2	Def. 2.6.5	Thm. 2.6.12	Thm. 2.6.14	Thm. 2.6.17
$\mathrm{BPA}^-_{\mathrm{drt}}$	Def. 2.7.2	Def. 2.7.6	Thm. 2.7.11	Thm. 2.7.13	Thm. 2.7.16
$\mathrm{BPA}^+_{\mathrm{drt}}$	Def. 2.8.2	Def. 2.8.4	Thm. 2.8.16	Thm. 2.8.18	Thm. 2.8.22
Concurrent Process Algebras					
$\mathrm{PA}^-_{\mathrm{drt}}\text{-ID}$	Def. 3.2.2	Def. 3.2.3	Cor. 3.2.9	Thm. 3.2.11	Thm. 3.2.16
$\mathrm{PA}^-_{\mathrm{drt}}\text{-ID}'$	Def. 3.3.2	Def. 3.3.3	Cor. 3.3.8	Cor. 3.3.17	Cor. 3.3.19
$\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID}$	Def. 3.4.3	Def. 3.4.5	Cor. 3.4.10	Thm. 3.4.12	Thm. 3.4.15
$\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID}'$	Def. 3.5.2	Def. 3.5.3	Cor. 3.5.8	Cor. 3.5.12	Cor. 3.5.14
$\mathrm{PA}^+_{\mathrm{drt}}$	Def. 3.6.2	Def. 3.6.3	Cor. 3.6.9	Thm. 3.6.11	Thm. 3.6.14
$\mathrm{ACP}^+_{\mathrm{drt}}$	Def. 3.7.2	Def. 3.7.3	Cor. 3.7.12	Thm. 3.7.14	Thm. 3.7.17

Table 44: Overview of definitions and theorems.

Please take note of the following:

- $\mathrm{PA}^-_{\mathrm{drt}}\text{-ID}$ and $\mathrm{PA}^-_{\mathrm{drt}}\text{-ID}'$ are two slightly different axiomatizations of the same theory with the same semantics, taken from [11] and [10] respectively. We prove both axiomatizations sound and complete.
- The same for $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID}$ and $\mathrm{ACP}^-_{\mathrm{drt}}\text{-ID}'$.

List of Tables

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